MATH 222 Homework

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Textbook: *Vector Calculus* by Jerrold E. Marsden and Anthony Tromba, Sixth Edition.

MATH 222 — Homework #1

Due January 18, 2023

Maxwell Lin

Problem 1

Use induction on k to prove that if $x_1, \ldots, x_k \in \mathbb{R}^n$, then

$$\|x_1 + \ldots + x_k\| \leq \|x_1\| + \ldots + \|x_k\|.$$

Solution

Proof.
Base case ($k = 1$): $\|x_1\| \leq \|x_1\|$ as required
Inductive step: Suppose $\|x_1 + \ldots + x_k\| \leq \|x_1\| + \ldots + \|x_k\|$. Then

$$\|x_1 + \ldots + x_k\| + \|x_{k+1}\| \leq \|x_1\| + \ldots + \|x_k\| + \|x_{k+1}\|$$

and

$$\|x_1 + \ldots + x_k + x_{k+1}\| \leq \|x_1 + \ldots + x_k\| + \|x_{k+1}\| \qquad \text{Triangle Inequality}$$

Thus,

$$\|x_1 + \ldots + x_k + x_{k+1}\| \leq \|x_1\| + \ldots + \|x_k\| + \|x_{k+1}\|$$

and $P(k) \implies P(k+1)$, completing the inductive step. Therefore, $\|x_1 + \ldots + x_k\| \leq \|x_1\| + \ldots + \|x_k\|$ for $\forall k \in \mathbb{Z}^+$.

$\square$

Problem 2

For any $x, y \in \mathbb{R}^n$,

$$\|\mathbf{x} - \mathbf{y}\| \geq \big|\|\mathbf{x}\| - \|\mathbf{y}\|\big|.$$

Solution

Proof.

$$\begin{aligned}
x \cdot y &\leq \|x\|\|y\| \qquad \text{Cauchy–Schwarz inequality} \\
-2x \cdot y &\geq -2\|x\|\|y\| \\
\|x\|^2 - 2x \cdot y + \|y\|^2 &\geq \|x\|^2 - 2\|x\|\|y\| + \|y\|^2 \\
\|x - y\|^2 &\geq (\|x\| - \|y\|)^2 \\
\|x - y\| &\geq \big|\|x\| - \|y\|\big|
\end{aligned}$$

$\square$

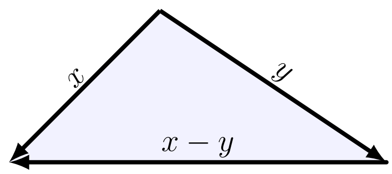


Figure 1: The length of any side of a triangle is greater than or equal to the absolute value of the difference of lengths of the other two sides.

Problem 3

For any natural number n, consider the following sets:

$$\begin{aligned} C^n &= \{\mathbf{x} \in \mathbb{R}^n \mid |x_i| \leq 1 \text{ for each } i = 1, \dots, n\} \\ D^n &= \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\} \\ D^n_r &= \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq r\} \quad \text{for any fixed } r > 0 \end{aligned}$$

Part A

Geometrically describe C^n and D^n for $n = 1, 2, 3$.

Solution

C^1 and D^1: The real number line on the interval [-1, 1]
C^2: All 2D points of a 2 by 2 square centered at the origin
D^2: All 2D points of the unit circle
C^3: All 3D points of a 2 by 2 by 2 cube centered at the origin
D^3: All 3D points of the unit sphere

Part B

Prove that $D^n \subset C^n$, but $D^n_r \not\subset C^n$ for any $r > 1$.

Solution

Proof. Suppose $x \in D^n$, then

$$\begin{aligned} \|x\| &\leq 1 \\ \sqrt{x_1^2 + \ldots + x_n^2} &\leq 1 \\ x_1^2 + \ldots + x_n^2 &\leq 1 \\ x_i &\leq \pm\sqrt{1 - (x_1^2 + \ldots + x_{i-1}^2 + x_{i+1}^2 + \ldots + x_n^2)} \\ |x_i| &\leq 1 \qquad \text{Since } 0 \leq x_1^2 + \ldots + x_n^2 \leq 1 \\ x &\in C^n \\ D^n &\subset C^n \qquad \text{as required.} \end{aligned}$$

For the second part of the proof, we have the counterexample: $re_1 \in D^n_r$, but $re_1 \notin C^n$ for $\forall r > 1$. $\square$

Part C

Prove that for any $\mathbf{x}, \mathbf{y} \in D^n$, $\|\mathbf{x} - \mathbf{y}\| \leq 2$.

Solution

Problem 3 continued on next page. . .

Proof. Suppose $x, y \in D^n$, then $\|x\| \leq 1$ and $\|y\| \leq 1$. This implies

$$\|x - y\| \leq \|x\| + \|-y\| = \|x\| + \|y\| \leq 2$$

□

Part D

Prove that we can find a pair of points $\mathbf{x}_n, \mathbf{y}_n \in C^n$ such that $\|\mathbf{x}_n - \mathbf{y}_n\| = 2\sqrt{n}$.

Proof. Take $x_n = (1, 1, \ldots, 1)$ and $y_n = (-1, -1, \ldots, -1)$. Then,

$$\begin{aligned} \|x_n - y_n\| &= \|(2, 2, \ldots, 2)\| \\ &= \sqrt{2^2 + 2^2 + \ldots + 2^2} \\ &= \sqrt{4n} \\ &= 2\sqrt{n} \end{aligned}$$

□

MATH 222 — Homework #2

Due January 25, 2023

Maxwell Lin

Problem 4

Prove that the set $A = \left\{(x, y) \in \mathbb{R}^2 \mid x > 0\right\}$ is open.

Solution

Proof. To show that A is open, we must show that there exists some $r > 0$ so that $D_r((x, y)) \subset A$ for $\forall (x, y) \in A$. We claim that $r = x$. Suppose $(u, v) \in D_r((x, y))$. Then

$$\begin{aligned} \|(u, v) - (x, y)\| &< r \\ \|(u - x, v - y)\| &< r \\ \sqrt{(u - x)^2 + (v - y)^2} &< r \\ |u - x| = \sqrt{(u - x)^2} \leq \sqrt{(u - x)^2 + (v - y)^2} &< r = x \end{aligned}$$

Which implies that

$$\begin{aligned} &\implies -x < u - x < x \\ &\implies 0 < u \\ &\implies (u, v) \in A \\ &\implies D_r((x, y)) \subset A \qquad \text{as required} \end{aligned}$$

□

Problem 5

Part A

If U and V are open sets, prove that $U \cup V$ and $U \cap V$ are open.

Solution

Proof. We have

$$U \cup V = \{x \in \mathbb{R}^n \mid x \in U \text{ or } x \in V\}$$

To prove that $U \cup V$ is open, we must show that for every $x \in U \cup V$ there exists a $r > 0$ such that $D_r(x) \subset (U \cup V)$. There are two cases.

Case 1: Suppose that $x \in U$. Since U is open, this implies that there exists $r > 0$ such that $D_r(x) \subset U$. Additionally, $U \subset (U \cup V)$ since any arbitrary element of U is also an element of $U \cup V$. Thus, if $x \in U$ we have $D_r(x) \subset (U \cup V)$.

Case 2: (We proceed the same way as Case 1.) Suppose that $x \in V$. Since V is open, this implies that there exists $r > 0$ such that $D_r(x) \subset V$. Additionally, $V \subset (U \cup V)$ since any arbitrary element of V is also an element of $U \cup V$. Thus, if $x \in V$ we have $D_r(x) \subset (U \cup V)$.

Therefore, $U \cup V$ is open.

For the second proof we have

$$U \cap V = \{x \in \mathbb{R}^n \mid x \in U \text{ and } x \in V\}$$

If $x \in (U \cap V)$, then there exists $r > 0$ such that $D_r(x) \subset U$ and a $s > 0$ such that $D_s(x) \subset V$. Either $r \leq s$ or $s \leq r$. Suppose that $r \leq s$. Thus, $D_r(x) \subset D_s(x) \subset V$ and therefore we have

$$D_r(x) \subset V \quad \text{and} \quad D_r(x) \subset U$$

which is equivalent to

$$D_r(x) \subset U \cap V.$$

Therefore, $U \cap V$ is open. □

Part B

More generally, if $U_1, U_2, U_3, \ldots$ are open sets in $\mathbb{R}^n$, prove that the infinite union $U = \bigcup_{i=1}^{\infty} U_i$ is open. (That is, $U = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \in U_i \text{ for some } i\}$.)

Solution

Proof. We proceed similarly to Part A. We need to show that for all $x \in U$, there exists an $r > 0$ such that $D_r(x) \subset U$.

If $x \in U$ then $x \in U_i$ for some i. Since U_i is an open set, there exists some $r_i > 0$ so that $D_{r_i}(x) \subset U_i$. Since all elements of U_i are elements of U, $U_i \subset U$ and we have $D_{r_i}(x) \subset U$ as required. □

Part C

Consider the sets $U_i = D_{1/i}(\mathbf{0})$ for $i = 1, 2, 3, \ldots$ Determine whether or not the infinite intersection $V = \bigcap_{i=1}^{\infty} U_i$ is open, and justify your answer. (That is, $V = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \in U_i \text{ for all } i\}$.)

Solution

Proof. First, let's examine what elements are in V.

For the sake of contradiction, suppose that there exists $x \neq 0 \in V$. Thus, $\|x\| > 0$. This means $x \in U_i$ for all i. However, there will always exist a $U_{\lceil \frac{1}{\|x\|} \rceil}$ which x is not contained in. In other words, there always exists a U_{i+1} that is a proper subset of U_i. This is a contradiction and therefore any nonzero x is not in V.

However, $x = 0 \in V$ since V is the infinite intersection of open disks with a positive radius. That is, $\|0\| \leq r$ for all $r > 0$.

Thus, the only element in V is the zero vector. There does not exist an $r > 0$ such that $D_r(0) \subset \{0\}$. For example, the point $(\frac{r}{2}, 0, \ldots, 0) \in D_r(0)$ but not in $\{0\}$. Thus, V is not an open set. □

MATH 222 — Homework #3

Due February 1, 2023

Maxwell Lin

Problem 1

Let $f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^6} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(a) Compute the limit as $(x,y) \to (0,0)$ of f along the path $x = 0$.
(b) Compute the limit as $(x,y) \to (0,0)$ of f along the path $x = y^3$.
(c) Show that f is not continuous at $(0,0)$.

Solution

For parts a and b we only need to examine the first condition of f where $(x,y) \neq (0,0)$ since the limit as $(x,y) \to (0,0)$ does not depend on the value of f at $(0,0)$.

a) If (x,y) approaches $(0,0)$ along the path $x = 0$, the limiting value is

$$\lim_{y\to 0} \frac{0y^3}{0^2 + y^6} = \lim_{y\to 0} 0 = 0$$

b) If (x,y) approaches $(0,0)$ along the path $x = y^3$, the limiting value is

$$\lim_{y\to 0} \frac{y^3y^3}{(y^3)^2 + y^6} = \lim_{y\to 0} \frac{y^6}{2y^6} = \frac{1}{2}$$

c) For f to be continuous at $(0,0)$, the $\lim_{(x,y)\to(0,0)} f$ must exist and equal $f(0,0)$. However, the limit does not exist since from parts a and b, we have two different paths to $(0,0)$ resulting in different limiting values. Thus for $\epsilon \leq \frac{1}{2}$, there is no $\delta > 0$ such that $\|x\| < \delta \implies |f(x)| < \epsilon$.

Problem 2

Compute the following limits if they exist:
(a) $\text{limit}_{(x,y)\to(0,0)} \frac{(x+y)^2-(x-y)^2}{xy}$
(b) $\text{limit}_{(x,y)\to(0,0)} \frac{\sin xy}{y}$
(c) $\text{limit}_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$

Solution

a)

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^2-(x-y)^2}{xy} = \lim_{(x,y)\to(0,0)} \frac{(x^2+2xy+y^2)-(x^2-2xy+y^2)}{xy} = \lim_{(x,y)\to(0,0)} 4 = 4$$

b)
$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{y} = \lim_{(x,y)\to(0,0)} \frac{x\sin(xy)}{xy} = \left(\lim_{(x,y)\to(0,0)} x\right)\left(\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy}\right) = (0)(1) = 0$$

To see why $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} = 1$, note that $\frac{\sin(xy)}{xy}$ can be rewritten as the composition of two functions: $(g \circ f)(x,y)$ where $f(x,y) = xy$ and $g(x) = \frac{sinx}{x}$. Since $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ and $\lim_{x\to 0} g(x) = 1$, $\lim_{(x,y)\to(0,0)}(f \circ g)(x) = 1$.

c) By the triangle inequality, we have

$$\left|\frac{x^3 - y^3}{x^2+y^2}\right| \le \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{-y^3}{x^2+y^2}\right| = \frac{|x|x^2}{x^2+y^2} + \frac{|y|y^2}{x^2+y^2}$$

Additionally, we have
$$\frac{|x|x^2}{x^2+y^2} \le |x|$$
since if $y = 0$ we have equality and if $y \neq 0$ the first expression is less than the second. Likewise,

$$\frac{|y|y^2}{x^2+y^2} \le |y|$$

Combining with the first inequality, we obtain

$$\left|\frac{x^3 - y^3}{x^2+y^2}\right| \le \frac{|x|x^2}{x^2+y^2} + \frac{|y|y^2}{x^2+y^2} \le |x| + |y|$$

Therefore,
$$-(|x|+|y|) \le \left|\frac{x^3 - y^3}{x^2+y^2}\right| \le |x| + |y|$$

By the squeeze theorem, since $\lim_{(x,y)\to(0,0)} -(|x|+|y|) = \lim_{(x,y)\to(0,0)} |x|+|y| = 0$

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2+y^2} = 0$$

Problem 3

Give the formulas for the partial derivatives at an arbitrary point (x, y) as well as at the specified values.
(a) $z = \sqrt{a^2 - x^2 - y^2}; (0,0), (a/2, a/2)$
(b) $z = \log\sqrt{1+xy}; (1,2), (0,0)$
(c) $z = e^{ax}\cos(bx+y); (2\pi/b, 0)$

Solution

a) $z = \sqrt{a^2 - x^2 - y^2}; (0,0), (a/2, a/2)$

$$\frac{\partial z}{\partial x}(x,y) = \frac{-x}{\sqrt{a^2-x^2-y^2}} \qquad \frac{\partial z}{\partial y}(x,y) = \frac{-y}{\sqrt{a^2-x^2-y^2}}$$

$$\frac{\partial z}{\partial x}(0,0) = 0 \quad \text{for } a \neq 0 \qquad \frac{\partial z}{\partial y}(0,0) = 0 \quad \text{for } a \neq 0$$

$$\frac{\partial z}{\partial x}(a/2,a/2) = \frac{-a\sqrt{2}}{|a|2} \quad \text{for } a \neq 0 \qquad \frac{\partial z}{\partial y}(a/2,a/2) = \frac{-a\sqrt{2}}{|a|2} \quad \text{for } a \neq 0$$

b) $z = \log\sqrt{1+xy}; (1,2), (0,0)$

$$\frac{\partial z}{\partial x}(x,y) = \frac{y}{2\sqrt{1+xy}\sqrt{1+xy}} = \frac{y}{2+2xy} \qquad \frac{\partial z}{\partial y}(x,y) = \frac{x}{2+2xy}$$
$$\frac{\partial z}{\partial x}(1,2) = \frac{2}{2+2(1)(2)} = \frac{1}{3} \qquad \frac{\partial z}{\partial y}(1,2) = \frac{1}{2+2(1)(2)} = \frac{1}{6}$$
$$\frac{\partial z}{\partial x}(0,0) = \frac{0}{2+2(0)(0)} = 0 \qquad \frac{\partial z}{\partial y}(0,0) = \frac{0}{2+2(0)(0)} = 0$$

c) $z = e^{ax}\cos(bx+y); (2\pi/b, 0)$

$$\frac{\partial z}{\partial x}(x,y) = ae^{ax}\cos(bx+y) - be^{ax}\sin(bx+y) \qquad \frac{\partial z}{\partial y}(x,y) = -e^{ax}\sin(bx+y)$$
$$\frac{\partial z}{\partial x}(2\pi/b, 0) = ae^{2\pi a/b} \qquad \frac{\partial z}{\partial y}(2\pi/b, 0) = 0$$

Problem 4

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by
$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$
Show that at $(x,y) = (0,0)$, f is "shmifferentiable" (i.e. its partial derivatives exist) but not continuous. This is another reason that merely knowing that the partials exist isn't a good definition of differentiability: we should expect, as in single-variable calculus, that differentiable implies continuous.

Solution

f is shmifferentiable at $(0,0)$ since
$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{(h)(0)}{h^2+0^2} - 0}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{(0)(h)}{0^2+h^2} - 0}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$
If f is continuous, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. However, the limit does not exist since if $(x,y) \to (0,0)$ along the path $x = y$, the limiting value is
$$\lim_{y\to 0}\frac{y^2}{2y^2} = \frac{1}{2}$$
while if $(x,y) \to (0,0)$ along the path $x = -y$ the limiting value is
$$\lim_{y\to 0}\frac{-y^2}{2y^2} = -\frac{1}{2}$$
Since approaching $(0,0)$ along different paths results in different limits, the limit does not exist and f is not continuous at $(0,0)$. That is, for $\epsilon \leq \frac{1}{2}$, there is no $\delta > 0$ such that $\|x\| < \delta \implies |f(x)| < \epsilon$.

Problem 5

Let $v > 0$ be a fixed real number. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,t) = e^{-(x-vt)^2}.$$

We are using t (rather than y) for the second variable because we are thinking of x as position and t as time. This is meant as a representation of a single wave traveling along a line at speed v.

(a) Draw the sections of the graph of f for $t = -1, 0, 1$, and explain why the above description makes sense. (Feel free to use a graphing device or look up "Gaussian function" to give you a basic sense of what the graph should look like.)

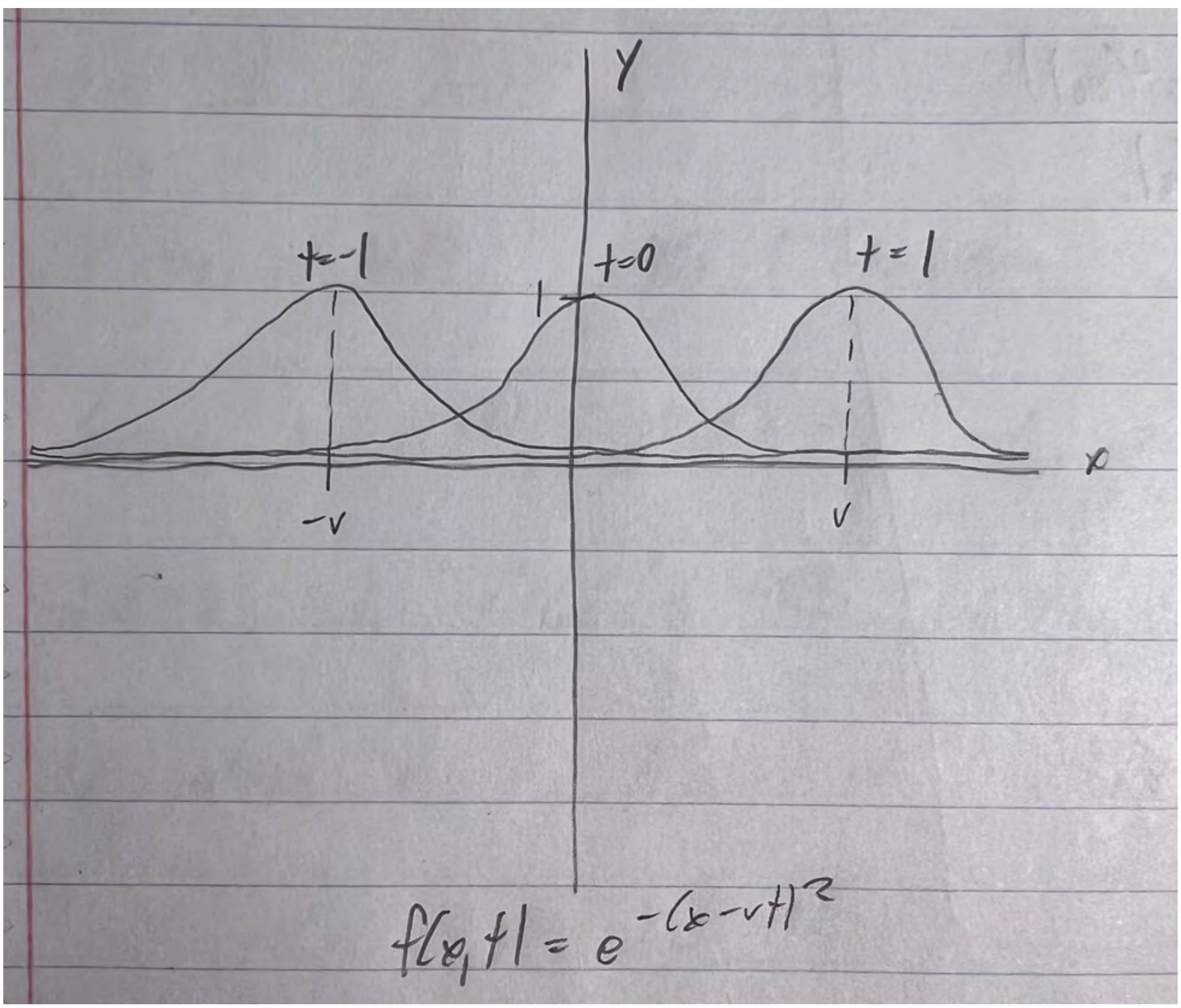
(b) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial t}$.

(c) Show that f satisfies the equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0.$$

(This is an example of a partial differential equation, a relation between the partial derivatives of a function of multiple variables. This particular equation is called the transport equation.)

Solution

a) 

The above description makes sense since e^{-x^2} represents a single wave at the origin. Multiplying speed with time results in distance which shifts the wave appropriately. That is, a positive time shifts the wave to the right and a negative time shifts the wave to the left. Increasing speed proportionally scales the rate at which the wave travels along the x-axis.

b)

$$\frac{\partial f}{\partial x} = -2(x - vt)(e^{-(x-vt)^2})$$
$$\frac{\partial f}{\partial t} = 2v(x - vt)(e^{-(x-vt)^2})$$

c)

$$\begin{aligned}\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} &= 2v(x - vt)(e^{-(x-vt)^2}) + v(-2(x - vt)(e^{-(x-vt)^2}))\\ &= 2v(x - vt)(e^{-(x-vt)^2}) - 2v(x - vt)(e^{-(x-vt)^2})\\ &= 0\end{aligned}$$

Problem 6

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. For any subset $U \subset \mathbb{R}^m$, we define $f^{-1}(U) = \{\mathbf{x} \in \mathbb{R}^n \mid f(x) \in U\}$ (that is, the set of point that map into U). Prove that f is continuous if and only if for every open set $U \subset \mathbb{R}^m$, $f^{-1}(U)$ is open.

Solution

Proof.
($\Longrightarrow$) Suppose $U \subset \mathbb{R}^n$ is an open set. We need to show that $f^{-1}(U)$ is open. Suppose $x_0 \in f^{-1}(U)$. By definition, $f(x_0) \in U$. Since U is open there exists $\epsilon > 0$ such that $D_\epsilon(f(x_0)) \subset U$. By continuity, $\forall\epsilon > 0$, there $\exists\delta > 0$ such that $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$. Thus, if $x \in D_\delta(x_0)$, we have $f(x) \in D_\epsilon(f(x_0)) \subset U$ and $x \in f^{-1}(U)$. Therefore, $D_\delta(x_0) \subset f^{-1}(U)$ for arbitrary $x_0 \in f^{-1}(U)$ proving that $f^{-1}(U)$ is an open set.

($\Longleftarrow$) Suppose $x_0 \in \mathbb{R}^n$ and $\epsilon > 0$. Since open balls are open sets, $D_\epsilon(f(x_0))$ is an open set in $\mathbb{R}^m$. By assumption, $U = f^{-1}(D_\epsilon(f(x_0)))$ is an open set in $\mathbb{R}^n$. Since $f(x_0) \in D_\epsilon(f(x_0))$, we have $x_0 \in U$. Since U is open, there is some $\delta > 0$ such that $D_\delta(x_0) \subset U = f^{-1}(D_\epsilon(f(x_0)))$. This means that if $x \in D_\delta(x_0)$, we have $x \in f^{-1}(D_\epsilon(f(x_0)))$ and $f(x) \in D_\epsilon(f(x_0))$. Equivalently, $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$ for arbitrary $x_0 \in \mathbb{R}^n$. Therefore, f is continuous. $\square$

MATH 222 — Homework #4

Due February 8, 2023

Maxwell Lin

Problem 1

Let $f(x, y) = e^{x^2 - y^2}$. Find the equation for the tangent plane to the graph of f at the point $(1, 1)$.

Solution

$$\begin{aligned} Df(1,1) &= \begin{bmatrix} \frac{\partial f}{\partial x}(1,1) & \frac{\partial f}{\partial y}(1,1) \end{bmatrix} \\ &= \begin{bmatrix} e^{x^2-y^2}(2x)\Big|_{(1,1)} & e^{x^2-y^2}(-2y)\Big|_{(1,1)} \end{bmatrix} \\ &= \begin{bmatrix} 2 & -2 \end{bmatrix} \end{aligned}$$

Thus, the tangent plane is parametrized by:

$$\begin{aligned} g(x,y) = f(1,1) + Df(1,1)((x,y) - (1,1)) &= 1 + \begin{bmatrix} 2 & -2 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix} \\ &= 1 + 2x - 2y \end{aligned}$$

Problem 2

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (xy, x + y)$.
(a) Compute the matrix of partial derivatives of f for an arbitrary point (x, y).
(b) Prove directly from the definition that f is differentiable at $(0, 0)$. (Do not use the theorem that C^1 implies differentiable.)

Solution

a)

$$Df(x,y) = \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix}$$

b) If f is differentiable at $(0, 0)$, then there exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - T(x - x_0)\|}{\|x - x_0\|} = 0$$

We proved in class that the matrix for T, if it exists, is the matrix of partial derivatives $Df(x, y)$. Thus,

$$\lim_{(x,y)\to(0,0)} \frac{\left\|(xy, x+y) - (0,0) - \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix}((x,y)-(0,0))\right\|}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{\|(-xy,0)\|}{\|(x,y)\|}$$
$$= \lim_{(x,y)\to(0,0)} \frac{|xy|}{\sqrt{x^2+y^2}}$$

We need to verify that $\lim_{(x,y)\to(0,0)} \frac{|xy|}{\sqrt{x^2+y^2}} = 0$. That is, we must show that for $\forall\varepsilon > 0$, there $\exists\delta > 0$ such that for all x where $0 < \|(x,y)-(0,0)\| < \delta$, $\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \varepsilon$. We claim that $\delta = \varepsilon$

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|x||y|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} < \delta = \varepsilon$$

The first inequality is reached since $|x| \leq \sqrt{x^2+y^2}$. (Equality if $y = 0$, less than if $y \neq 0$).

Thus, the limit is verified and f is differentiable at $(0,0)$.

Problem 3

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

(a) Using the results of the previous homework, show that f is continuous at $(0,0)$

(b) For an arbitrary nonzero vector $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, compute $D_{\vec{v}}f(0,0)$.

(c) Is f differentiable at $(0,0)$? Explain why or why not.

(d) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(e) Show directly from (d) that $\frac{\partial f}{\partial x}$ is not continuous at $(0,0)$. (Hint: Consider the limits along the lines $x = 0$ and $y = 0$.)

Solution

a) If f is continuous at $(0,0)$, $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$. From HW 3, we have that

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0 = f(0,0)$$

as required.

Problem 3 continued on next page...

b)

$$\begin{aligned}
D_{\vec{v}}f(0,0) &= \lim_{h\to 0}\frac{f((0,0)+h(a,b))-f(0,0)}{h}\\
&= \lim_{h\to 0}\frac{\left[\frac{(ha)^3-(hb)^3}{(ha)^2+(hb)^2}\right]-0}{h}\\
&= \lim_{h\to 0}\frac{h^3(a^3-b^3)}{h^3(a^2+b^2)}\\
&= \lim_{h\to 0}\frac{a^3-b^3}{a^2+b^2}\\
&= \frac{a^3-b^3}{a^2+b^2}
\end{aligned}$$

c) f is not differentiable at $(0,0)$. For f to be differentiable, $D_{\vec{v}}f(0,0)$ must exist for $\forall\vec{v}$ and $\vec{v}\to D_{\vec{v}}f(0,0)$ must be linear. However, $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \frac{a^3-b^3}{a^2+b^2}$ is not linear since

$$D_v f(0,0) = \frac{a^3-b^3}{a^2+b^2} \neq a-b = \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = Df(0,0)(v).$$

For instance, take $(a,b)=(2,1)$, and we can see

$$\frac{2^3-1^3}{2^2+1^2} = \frac{7}{5} \neq 1.$$

d) First we compute $\frac{\partial f}{\partial x}$. When $(x,y)\neq(0,0)$,

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= \frac{3x^2(x^2+y^2)-2x(x^3-y^3)}{(x^2+y^2)^2}\\
&= \frac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2}
\end{aligned}$$

When $(x,y)=(0,0)$,

$$\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}\\
&= \lim_{h\to 0}\frac{\frac{h^3-0^3}{h^2+0^2}-0}{h}\\
&= \lim_{h\to 0}\frac{h}{h}\\
&= 1
\end{aligned}$$

Thus,

$$\frac{\partial f}{\partial x} = \begin{cases} \frac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2} & (x,y)\neq(0,0)\\ 1 & (x,y)=(0,0)\end{cases}$$

Now we compute $\frac{\partial f}{\partial y}$. When $(x,y)\neq(0,0)$,

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= \frac{-3y^2(x^2+y^2)-2y(x^3-y^3)}{(x^2+y^2)^2} \\ &= \frac{-y^4-3y^2x^2-2yx^3}{(x^2+y^2)^2}\end{aligned}$$

When $(x,y)=(0,0)$,

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} \\ &= \lim_{h\to 0}\frac{\frac{0^3-h^3}{0^2+h^2}-0}{h} \\ &= \lim_{h\to 0}\frac{-h}{h} \\ &= -1\end{aligned}$$

Thus,

$$\frac{\partial f}{\partial y} = \begin{cases}\frac{-y^4-3y^2x^2-2yx^3}{(x^2+y^2)^2} & (x,y)\neq(0,0) \\ -1 & (x,y)=(0,0)\end{cases}$$

e) We have

$$\frac{\partial f}{\partial x} = \begin{cases}\frac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2} & (x,y)\neq(0,0) \\ 1 & (x,y)=(0,0)\end{cases}$$

If (x,y) approaches $(0,0)$ along the path $x=0$, the limiting value is

$$\lim_{y\to 0}\frac{0}{y^4}=0$$

If (x,y) approaches $(0,0)$ along the path $y=0$, the limiting value is

$$\lim_{x\to 0}\frac{x^4}{x^4}=1$$

Since different paths to $(0,0)$ have different limits, $\lim_{(x,y)\to(0,0)}\frac{\partial f}{\partial x}$ does not exist and therefore, $\frac{\partial f}{\partial x}$ is not continuous at $(0,0)$.

Problem 4

Consider the function $f:\mathbb{R}\to\mathbb{R}$ given by

$$f(x) = \begin{cases}x^2\sin\left(\frac{1}{x}\right) & x\neq 0 \\ 0 & x=0.\end{cases}$$

(a) Prove that f is differentiable everywhere, and give a formula for f'. (Hint: To compute $f'(0)$, use the limit definition of the derivative.)
(b) Prove that f' is not continuous at $x=0$, and thus that f is not C^1.

Solution

Problem 4 continued on next page...

a) For $x \neq 0$, we have

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

For $x = 0$, we have

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2\sin\left(\frac{1}{h}\right)-0}{h} = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right)$$

Since $-h \leq h\sin\left(\frac{1}{h}\right) \leq h$ and $\lim_{h\to 0}(-h) = \lim_{h\to 0}(h) = 0$, we have by the squeeze theorem that $\lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$.

Thus,

$$f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and f is differentiable everywhere.

b) Suppose that f' was continuous at $x = 0$ so that $\lim_{x\to 0}\left[2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right]$ exists and equals 0. We also know that $\lim_{x\to 0}\left[2x\sin\left(\frac{1}{x}\right)\right]$ exists and equals 0 since we can squeeze this function between $-2x$ and $2x$. Thus,

$$\begin{aligned} 0 &= \lim_{x\to 0}\left[2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right] - \lim_{x\to 0}\left[2x\sin\left(\frac{1}{x}\right)\right] \\ &= \lim_{x\to 0}\left[2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) - 2x\sin\left(\frac{1}{x}\right)\right] \\ &= \lim_{x\to 0}\left[-\cos\left(\frac{1}{x}\right)\right]. \end{aligned}$$

However, this is a contradiction since we know that $\lim_{x\to 0}\left[-\cos\left(\frac{1}{x}\right)\right]$ does not exist. (This limit is equivalent to $\lim_{x\to\infty}\left[-\cos(x)\right]$ which does not exist since cos oscillates between -1 and 1 as $x \to \infty$.)

Therefore, f' is not continuous at $x = 0$ and thus, f is not C^1.

Problem 5

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Prove that f is C^1 everywhere.

Solution

First we compute $\frac{\partial f}{\partial x}$. If $(x,y) \neq (0,0)$, we have

$$\frac{\partial f}{\partial x}(x,y) = \frac{2xy^4}{(x^2+y^2)^2}$$

Problem 5 continued on next page...

If $(x, y) = (0, 0)$, we have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{0-0}{h}$$
$$= 0$$

Thus,

$$\frac{\partial f}{\partial x} = \begin{cases} \frac{2xy^4}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

We know that $\frac{\partial f}{\partial x}$ is continuous for all $(x, y) \neq (0, 0)$. We must check if $\frac{\partial f}{\partial x}$ is continuous at $(0, 0)$.
We must show that $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = 0$. This means that we must show that for $\forall \varepsilon > 0$, there $\exists \delta > 0$ such that for all (x, y) where $0 < \|(x, y) - (0, 0)\| < \delta$, $\left|\frac{2xy^4}{(x^2+y^2)^2} - 0\right| < \varepsilon$. We claim that $\delta = \frac{\varepsilon}{2}$.

$$\left|\frac{2xy^4}{(x^2+y^2)^2} - 0\right| \leq \frac{2(x^2+y^2)^{5/2}}{(x^2+y^2)^2} = 2\sqrt{x^2+y^2} < 2\delta = \varepsilon$$

as required.

Now we compute $\frac{\partial f}{\partial y}$. If $(x, y) \neq (0, 0)$, we have

$$\frac{\partial f}{\partial x}(x,y) = \frac{2yx^4}{(x^2+y^2)^2}$$

If $(x, y) = (0, 0)$, we have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{0-0}{h}$$
$$= 0$$

Thus,

$$\frac{\partial f}{\partial x} = \begin{cases} \frac{2yx^4}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

We know that $\frac{\partial f}{\partial y}$ is continuous for all $(x, y) \neq (0, 0)$. We must check if $\frac{\partial f}{\partial y}$ is continuous at $(0, 0)$.
We must show that $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y} = 0$. This means that we must show that for $\forall \varepsilon > 0$, there $\exists \delta > 0$ such that for all (x, y) where $0 < \|(x, y) - (0, 0)\| < \delta$, $\left|\frac{2yx^4}{(x^2+y^2)^2} - 0\right| < \varepsilon$. We claim that $\delta = \frac{\varepsilon}{2}$

$$\left|\frac{2yx^4}{(x^2+y^2)^2} - 0\right| \leq \frac{2(x^2+y^2)^{5/2}}{(x^2+y^2)^2} = 2\sqrt{x^2+y^2} < 2\delta = \varepsilon$$

as required.

Since all partial derivatives of f exist and are continuous everyone, f is C^1 everywhere.

MATH 222 — Homework #5

Due February 15, 2023

Maxwell Lin

Problem 1

Section 2.4, #6.
Give a parametrization for each of the following curves:
(a) The line passing through $(1, 2, 3)$ and $(-2, 0, 7)$
(b) The graph of $f(x) = x^2$
(c) The square with vertices $(0, 0), (0, 1), (1, 1)$, and $(1, 0)$ (Break it up into line segments.)
(d) The ellipse given by $\frac{x^2}{9} + \frac{y^2}{25} = 1$

Solution

a) $l\colon \mathbb{R} \to \mathbb{R}^3$ given by
$$l(t) = (1, 2, 3) + (-3, -2, 4)t = (1 - 3t, 2 - 2t, 3 + 4t)$$

b) $c\colon \mathbb{R} \to \mathbb{R}^2$ given by
$$c(t) = (t, t^2)$$

c) $c\colon [0, 4) \to \mathbb{R}^2$ given by
$$c(t) = \begin{cases} (0, t) & 0 \le t < 1 \\ (t - 1, 1) & 1 \le t < 2 \\ (1, -t + 3) & 2 \le t < 3 \\ (-t + 4, 0) & 3 \le t < 4 \end{cases}$$

d) $c\colon [0, 2\pi) \to \mathbb{R}^2$ given by
$$c(t) = (3\cos(t), 5\sin(t))$$

Problem 2

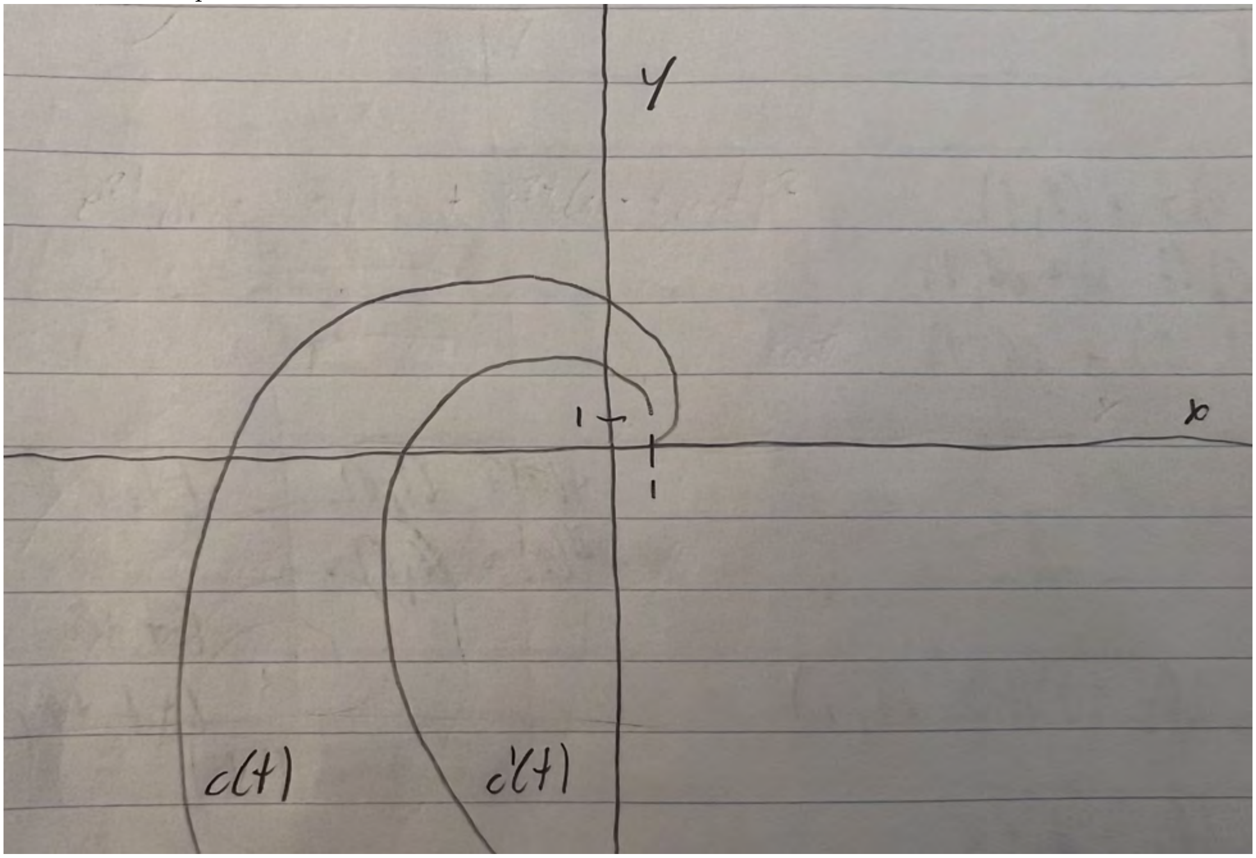
Section 2.4, #24.
Consider the spiral given by $\mathbf{c}(t) = (e^t \cos(t), e^t \sin(t))$. Show that the angle between $\mathbf{c}$ and $\mathbf{c}'$ is constant. What is the angle between c and c'? Draw this curve.

Solution

Taking the 1st derivative gives us
$$\begin{aligned} c'(t) &= (e^t \cos(t) - e^t \sin(t), e^t \sin(t) + e^t \cos(t)) \\ &= (e^t(\cos(t) - \sin(t)), e^t(\sin(t) + \cos(t))). \end{aligned}$$

Thus, the angle between c and c' is
$$\theta = \cos^{-1}\left(\frac{c(t) \cdot c'(t)}{\|c(t)\|\|c'(t)\|}\right) = \cos^{-1}\left(\frac{e^{2t}}{\sqrt{e^{2t}}\sqrt{2e^{2t}}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \boxed{\frac{\pi}{4}}$$

a constant as required.



Problem 3

Section 2.5, #8.
Let $f(u,v,w) = (e^{u-w}, \cos(v+u) + \sin(u+v+w))$ and $g(x,y) = (e^x, \cos(y-x), e^{-y})$. Calculate $f \circ g$ and $\mathbf{D}(f \circ g)(0,0)$. Also compute Df and Dg at the relevant points and verify that the chain rule holds.

Solution

The function composition is

$$(f \circ g)(x,y) = (e^{e^x - e^{-y}}, \cos(\cos(y-x) + e^x) + \sin(e^x + \cos(y-x) + e^{-y})).$$

The derivative of $f \circ g$ at $(0,0)$ is

$$\begin{aligned} \mathbf{D}(f \circ g)(0,0) &= \left. \begin{bmatrix} e^{e^x - e^{-y}}(e^x) & e^{e^x - e^{-y}}(e^{-y}) \\ \frac{\partial (f \circ g)_2}{\partial x} & \frac{\partial (f \circ g)_2}{\partial y} \end{bmatrix} \right|_{(0,0)} \\ &= \begin{bmatrix} 1 & 1 \\ -\sin(2) + \cos(3) & -\cos(3) \end{bmatrix} \end{aligned}$$

where

$$\frac{\partial (f \circ g)_2}{\partial x} = -\sin(\cos(y-x) + e^x)(\sin(y-x) + e^x) + \cos(e^x + \cos(y-x) + e^{-y})(e^x + \sin(y-x))$$

Problem 3 continued on next page...

$$\frac{\partial(f\circ g)_2}{\partial y} = -\sin(\cos(y-x)+e^x)(-\sin(y-x)) + \cos(e^x+\cos(y-x)+e^{-y})(-\sin(y-x)-e^{-y})$$

The chain rule states that

$$\begin{aligned}\mathbf{D}(f\circ g)(0,0) &= \mathbf{D}f(g(0,0))\mathbf{D}g(0,0)\\ &= \mathbf{D}f(1,1,1)\mathbf{D}g(0,0) \qquad \text{Since } g(0,0)=(1,1,1)\end{aligned}$$

We have

$$\begin{aligned}\mathbf{D}f(1,1,1) &= \left.\begin{bmatrix} e^{u-w} & 0 & -e^{u-w}\\ -\sin(v+u)+\cos(u+v+w) & -\sin(v+u)+\cos(u+v+w) & \cos(u+v+w)\end{bmatrix}\right|_{(1,1,1)}\\ &= \begin{bmatrix} 1 & 0 & -1\\ -\sin(2)+\cos(3) & -\sin(2)+\cos(3) & \cos(3)\end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathbf{D}g(0,0) &= \left.\begin{bmatrix} e^x & 0\\ \sin(y-x) & -\sin(y-x)\\ 0 & -e^-y\end{bmatrix}\right|_{(0,0)}\\ &= \begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & -1\end{bmatrix}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{D}(f\circ g)(0,0) &= \begin{bmatrix} 1 & 0 & -1\\ -\sin(2)+\cos(3) & -\sin(2)+\cos(3) & \cos(3)\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & -1\end{bmatrix}\\ &= \begin{bmatrix} 1 & 1\\ -\sin(2)+\cos(3) & -\cos(3)\end{bmatrix}\end{aligned}$$

and the chain rule holds.

Problem 4

Section 2.5, #12.
Let $h\colon \mathbb{R}^3 \to \mathbb{R}^5$ and $g\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by $h(x,y,z) = \left(xyz, e^{xz}, x\sin(y), \frac{-9}{x}, 17\right)$ and $g(u,v) = \left(v^2+2u, \pi, 2\sqrt{u}\right)$. Find $\mathbf{D}(h\circ g)(1,1)$.

Solution

We proceed by the chain rule which states $\mathbf{D}(h\circ g)(1,1) = \mathbf{D}h(g(1,1))\mathbf{D}g(1,1)$. We have

$$\begin{aligned}\mathbf{D}h(g(1,1)) = \mathbf{D}h(3,\pi,2) &= \left.\begin{bmatrix} yz & xz & xy\\ ze^{xz} & 0 & xe^{xz}\\ \sin(y) & x\cos(y) & 0\\ \frac{9}{x^2} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right|_{(3,\pi,2)}\\ &= \begin{bmatrix} 2\pi & 6 & 3\pi\\ 2e^6 & 0 & 3e^6\\ 0 & 3 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\end{aligned}$$

Problem 4 continued on next page. . .

and

$$\begin{aligned}\mathbf{D}g(1,1) &= \left.\begin{bmatrix} 2 & 2v \\ 0 & 0 \\ u^{-1/2} & 0 \end{bmatrix}\right|_{(1,1)} \\ &= \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{D}(h \circ g)(1,1) &= \begin{bmatrix} 2\pi & 6 & 3\pi \\ 2e^6 & 0 & 3e^6 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 7\pi & 4\pi \\ 7e^6 & 4e^6 \\ 0 & 0 \\ 2 & 2 \\ 0 & 0 \end{bmatrix}\end{aligned}$$

Problem 5

Section 2.5, #22.
This exercise gives another example of the fact that the chain rule is not applicable if f is not differentiable. Consider the function

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Show that
(a) $\partial f/\partial x$ and $\partial f/\partial y$ exist at $(0,0)$.
(b) If $\mathbf{g}(t) = (at, bt)$ for constants a and b, then $f \circ \mathbf{g}$ is differentiable and $(f \circ \mathbf{g})'(0) = ab^2/\left(a^2+b^2\right)$, but $\nabla f(0,0) \cdot \mathbf{g}'(0) = 0$.

Solution

(a) We have

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{0-0}{h} \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{0-0}{h} \\ &= 0.\end{aligned}$$

Thus, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$.

(b) The composition of g with f is

$$\begin{aligned}(f\circ g)(t) &= \begin{cases}\frac{ab^2t}{a^2+b^2} & t\neq 0\\ 0 & t=0\end{cases}\\ &= \frac{ab^2t}{a^2+b^2}\end{aligned}$$

Thus, $f\circ g$ is differentiable where

$$(f\circ g)'(t) = \frac{ab^2}{a^2+b^2}$$

However,

$$\nabla f(0,0)\cdot g'(0) = \begin{bmatrix}0\\0\end{bmatrix}\cdot\begin{bmatrix}a\\b\end{bmatrix} = 0$$

Thus, the chain rule fails to hold, and we deduce that f cannot have been differentiable at $(0,0)$.

Problem 6

Section 4.2, #8.
Recall from Section 2.4 that a rolling circle of radius R traces out a cycloid, which can be parametrized by $\mathbf{c}(t) = (Rt - R\sin t, R - R\cos t)$. One arch of the cycloid is completed from $t=0$ to $t=2\pi$. Show that the length of this arch is always 4 times the diameter of the rolling circle.

Solution

The length of the arch is

$$\begin{aligned}\int_0^{2\pi} \|c'(t)\|\,dt &= \int_0^{2\pi} \|(R - R\cos(t), R\sin(t))\|\,dt\\ &= \int_0^{2\pi} \sqrt{(R-R\cos(t))^2 + (R\sin(t))^2}\,dt\\ &= \int_0^{2\pi} \sqrt{R^2 - 2R^2\cos(t) + R^2\cos^2(t) + R^2\sin^2(t)}\,dt\\ &= \int_0^{2\pi} \sqrt{R^2(1 - 2\cos(t) + \cos^2(t) + \sin^2(t))}\,dt\\ &= \int_0^{2\pi} \sqrt{R^2(2-2\cos(t))}\,dt\\ &= R\sqrt{2}\int_0^{2\pi} \sqrt{1-\cos(t)}\,dt\\ &= 2R\int_0^{2\pi} \sin\left(\frac{t}{2}\right)dt \qquad \text{Since } \sin\left(\frac{t}{2}\right) = \sqrt{\frac{1-\cos(t)}{2}}\\ &= -4R\left[\cos\left(\frac{t}{2}\right)\right]_0^{2\pi}\\ &= -4R\,[-1-1]\\ &= 8R\end{aligned}$$

which is 4 times the diameter $2R$ as required.

Problem 7

Let $\mathbf{c} : [a, b] \to \mathbb{R}^n$ be a continuously differentiable path, and let $f : [p, q] \to [a, b]$ be a continuously differentiable function with the property that $f'(t) \geq 0$ for all $t \in [p, q]$. Prove that the arc lengths of $\mathbf{c}$ and $\mathbf{c} \circ f$ are equal.

Solution

Proof. To prove that the arc lengths of c and $(c \circ f)$ are equal we must show that $\int_a^b \|c'(t)\| \, dt = \int_p^q \|(c \circ f)(t)\| \, dt$.

$$
\begin{aligned}
\int_p^q \|(c \circ f)(t)\| \, dt &= \int_p^q \|c'(f(t))f'(t)\| \, dt \qquad \text{Chain rule} \\
&= \int_p^q \|c'(f(t))\| |f'(t)| \, dt \\
&= \int_p^q \|c'(f(t))\| f'(t) \, dt \qquad \text{Since } f'(t) \geq 0 \\
&= \int_{f(p)}^{f(q)} \|c'(u)\| \, du \qquad \text{Let } u = f(t) \\
&= \int_a^b \|c'(u)\| \, du
\end{aligned}
$$

□

MATH 222 — Homework #6

Due March 2, 2023

Maxwell Lin

Problem 1

Section 3.1, #12. (You can read about the physical significance of the heat equation on page 154.)

(a) Show that $T(x,t) = e^{-kt}\cos x$ satisfies the one-dimensional heat equation

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}.$$

(b) Show that $T(x,y,t) = e^{-kt}(\cos x + \cos y)$ satisfies the two-dimensional heat equation

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \frac{\partial T}{\partial t}.$$

(c) Show that $T(x,y,z,t) = e^{-kt}(\cos x + \cos y + \cos z)$ satisfies the three-dimensional heat equation

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \frac{\partial T}{\partial t}$$

Solution

(a) We have

$$\frac{\partial T}{\partial t} = -ke^{-kt}\cos x$$

and

$$\begin{aligned}
\frac{\partial T}{\partial x} &= -e^{-kt}\sin x \\
\frac{\partial^2 T}{\partial x^2} &= -e^{-kt}\cos x \\
k\frac{\partial^2 T}{\partial x^2} &= -ke^{-kt}\cos x.
\end{aligned}$$

(b) We have

$$\frac{\partial T}{\partial t} = -ke^{-kt}(\cos x + \cos y)$$

and

$$\begin{aligned}
\frac{\partial T}{\partial x} &= -e^{-kt}\sin x & \qquad \frac{\partial T}{\partial y} &= -e^{-kt}\sin y \\
\frac{\partial^2 T}{\partial x^2} &= -e^{-kt}\cos x & \qquad \frac{\partial^2 T}{\partial y^2} &= -e^{-kt}\cos y
\end{aligned}$$

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = -ke^{-kt}(\cos x + \cos y).$$

(c) We have

$$\frac{\partial T}{\partial t} = -ke^{-kt}(\cos x + \cos y + \cos z)$$

and

$$\begin{aligned}
\frac{\partial T}{\partial x} &= -e^{-kt}\sin x & \frac{\partial T}{\partial y} &= -e^{-kt}\sin y & \frac{\partial T}{\partial z} &= -e^{-kt}\sin z \\
\frac{\partial^2 T}{\partial x^2} &= -e^{-kt}\cos x & \frac{\partial^2 T}{\partial y^2} &= -e^{-kt}\cos y & \frac{\partial^2 T}{\partial z^2} &= -e^{-kt}\cos z
\end{aligned}$$

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = -ke^{-kt}(\cos x + \cos y + \cos z).$$

Problem 2

Section 3.2, #4.

Determine the second-order Taylor formula for the given function about the given point (x_0, y_0).

$$f(x,y) = 1/\left(x^2+y^2+1\right), \quad \text{where } x_0 = 0,\ y_0 = 0$$

Solution

The second-order Taylor formula for $f(x,y)$ at $(0,0)$ is given by

$$f(h) = f(0,0) + \mathbf{D}f(0,0)\begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix}\mathbf{H}f(0,0)\begin{bmatrix} x \\ y \end{bmatrix} + R_2(0,h)$$

where

$$\lim_{h\to 0}\frac{R_2(0,h)}{\|h\|^2} = 0.$$

We have

$$\begin{aligned}
\mathbf{D}f(0,0) &= \left.\begin{bmatrix} \dfrac{-2x}{(x^2+y^2+1)^2} & \dfrac{-2y}{(x^2+y^2+1)^2} \end{bmatrix}\right|_{(0,0)} \\
&= \begin{bmatrix} 0 & 0 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{H}f(0,0) &= \left.\begin{bmatrix} \dfrac{2(3x^2-y^2-1)}{(x^2+y^2+1)^3} & \dfrac{8xy}{(x^2+y^2+1)^3} \\ \dfrac{8xy}{(x^2+y^2+1)^3} & \dfrac{2(3y^2-x^2-1)}{(x^2+y^2+1)^3} \end{bmatrix}\right|_{(0,0)} \\
&= \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}
\end{aligned}$$

Thus, we obtain

$$f(h) = 1 - x^2 - y^2 + R_2(0,h).$$

Problem 3

Section 3.3, #6.

Find the critical points of the given function and then determine whether they are local maxima, local minima, or saddle points.

$$f(x, y) = x^2 - 3xy + 5x - 2y + 6y^2 + 8$$

Solution

The critical points occur when

$$\nabla f(x, y) = \begin{bmatrix} 2x - 3y + 5 \\ -3x - 2 + 12y \end{bmatrix} = 0$$

which only occurs at the point $(\frac{-18}{5}, \frac{-11}{15})$. The Hessian at $(\frac{-18}{5}, \frac{-11}{15})$ is

$$\mathbf{H}f\left(\frac{-18}{5}, \frac{-11}{15}\right) = \begin{bmatrix} 2 & -3 \\ -3 & 12 \end{bmatrix}.$$

Since $\det(\mathbf{H}f(x, y)) = 15 > 0$ and $a = 2 > 0$, we have that $(\frac{-18}{5}, \frac{-11}{15})$ is a local minimum.

Problem 4

Section 3.3, #10.

Find the critical points of the given function and then determine whether they are local maxima, local minima, or saddle points.

$$f(x, y) = y + x \sin y$$

Solution

The critical points occur when

$$\nabla f(x, y) = \begin{bmatrix} \sin y \\ 1 + x \cos y \end{bmatrix} = 0.$$

We have that $\sin y = 0$ whenever $y = k\pi$ for $k \in \mathbb{Z}$. For x, there are two cases. If k is an even integer, then $\cos y = \cos k\pi = 1$ and x must be -1. If k is an odd integer, then $\cos y = \cos k\pi = -1$ and x must be 1. Thus, the critical points of $f(x, y)$ are all points of the form

$$(x, y) = \begin{cases} (-1, k\pi) & k \text{ even integer} \\ (1, k\pi) & k \text{ odd integer.} \end{cases}$$

The Hessian is

$$\mathbf{H}f(x, y) = \begin{bmatrix} 0 & \cos y \\ \cos y & -x \sin y \end{bmatrix}.$$

Since the period of sin and cos is 2π, there are only two cases. When k is even, we have

$$\mathbf{H}f(-1, k\pi) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

When k is odd, we have

$$\mathbf{H}f(1, k\pi) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

In both cases, the determinant is $-1 < 0$. Therefore, all critical points are saddle points.

Problem 5

Section 3.3, #18.

Let $f(x, y, z) = x^2 + y^2 + z^2 + kyz$.
(a) Verify that $(0, 0, 0)$ is a critical point for f.
(b) Find all values of k such that f has a local minimum at $(0, 0, 0)$.

Solution

(a) We have

$$\begin{aligned} \nabla f(0,0,0) &= \left.\begin{bmatrix} 2x \\ 2y + kz \\ 2z + ky \end{bmatrix}\right|_{(0,0,0)} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

as required.

(b) The Hessian of f at $(0, 0, 0)$ is

$$\mathbf{H}f(0,0,0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & k \\ 0 & k & 2 \end{bmatrix}.$$

For $\mathbf{H}f(0, 0, 0)$ to be positive-definite, the determinants of all the diagonal submatrices must be greater than 0. We already have

$$\det\left(\begin{bmatrix} 2 \end{bmatrix}\right) > 0$$

and

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) > 0.$$

Thus, the only constraint is

$$\begin{gathered} \det\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & k \\ 0 & k & 2 \end{bmatrix}\right) = 8 - 2k^2 > 0 \\ -2 < k < 2 \end{gathered}$$

We must also check when the determinant is 0 which is when $k = \pm 2$. We then have

$$x^2 + y^2 + z^2 + 2yz = x^2 + (y + z)^2$$

which is positive for all $(x, y, z) \neq 0$. Therefore, f has a local minimum at $(0, 0, 0)$ for all

$$-2 \leq k \leq 2.$$

Problem 6

Section 3.3, #36.

Let n be an integer greater than 2 and set $f(x, y) = ax^n + cy^n$, where $ac \neq 0$. Determine the nature of the critical points of f.

Solution

The critical points occur when

$$\nabla f(x, y) = \begin{bmatrix} anx^{n-1} \\ cny^{n-1} \end{bmatrix} = 0.$$

Thus, $(0, 0)$ is the only critical point.

Now suppose n is even. If a and c are both positive, then $(0, 0)$ is a local minimum since $f > 0$ for all $(x, y) \neq 0$. If a and c are both negative, then $(0, 0)$ is a local maximum since $f < 0$ for all $(x, y) \neq 0$. If a and c have different signs, then $(0, 0)$ is a saddle point since it is neither a local minimum or maximum. (For example, suppose a was negative and c was positive. If we observe the cross-section $x = 0$, then $(0, 0)$ would be a local minimum. If instead, we observe the cross-section $y = 0$, $(0, 0)$ would be a local maximum. Thus, there is no neighborhood containing $(0, 0)$ so that all $f(x, y) \geq 0$ or all $f(x, y) \leq 0$. Therefore, $(0, 0)$ is a saddle point.)

Now suppose n is odd. Regardless of a and c, $(0, 0)$ is a saddle point since there always exists values of (x, y) near $(0, 0)$ such that f is greater than 0 and less than 0. For example, along the path $y = 0$, $f < 0$ for all $x < 0$ and $f > 0$ for all $x > 0$. (This is assuming a was positive. If a was negative, $f > 0$ for all $x < 0$ and $f < 0$ for all $x > 0$.)

Problem 7

Consider the quadratic form $f(x, y) = 5x^2 - 8xy + 5y^2$.

(a) Find the symmetric matrix A such that $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

(b) Find an orthonormal basis $(\mathbf{v}_1, \mathbf{v}_2)$ for $\mathbb{R}^2$ consisting of eigenvectors for A, along with their corresponding eigenvalues.

(c) Express the function f using $(\mathbf{v}_1, \mathbf{v}_2)$ coordinates.

(d) Draw the level sets $f(x, y) = c$ for $c = -1, 0, 1$. (Hint: It helps to draw a second set of axes corresponding to the orthonormal basis of eigenvectors, and then draw the level sets with respect to these axes.)

(e) Suppose $g : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function, and for some critical point $\mathbf{x}_0$ of g, the Hessian of g at $\mathbf{x}_0$ is equal to the matrix A. Does g have a local minimum, local maximum, or saddle point at $\mathbf{x}_0$?

Solution

(a)

$$A = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

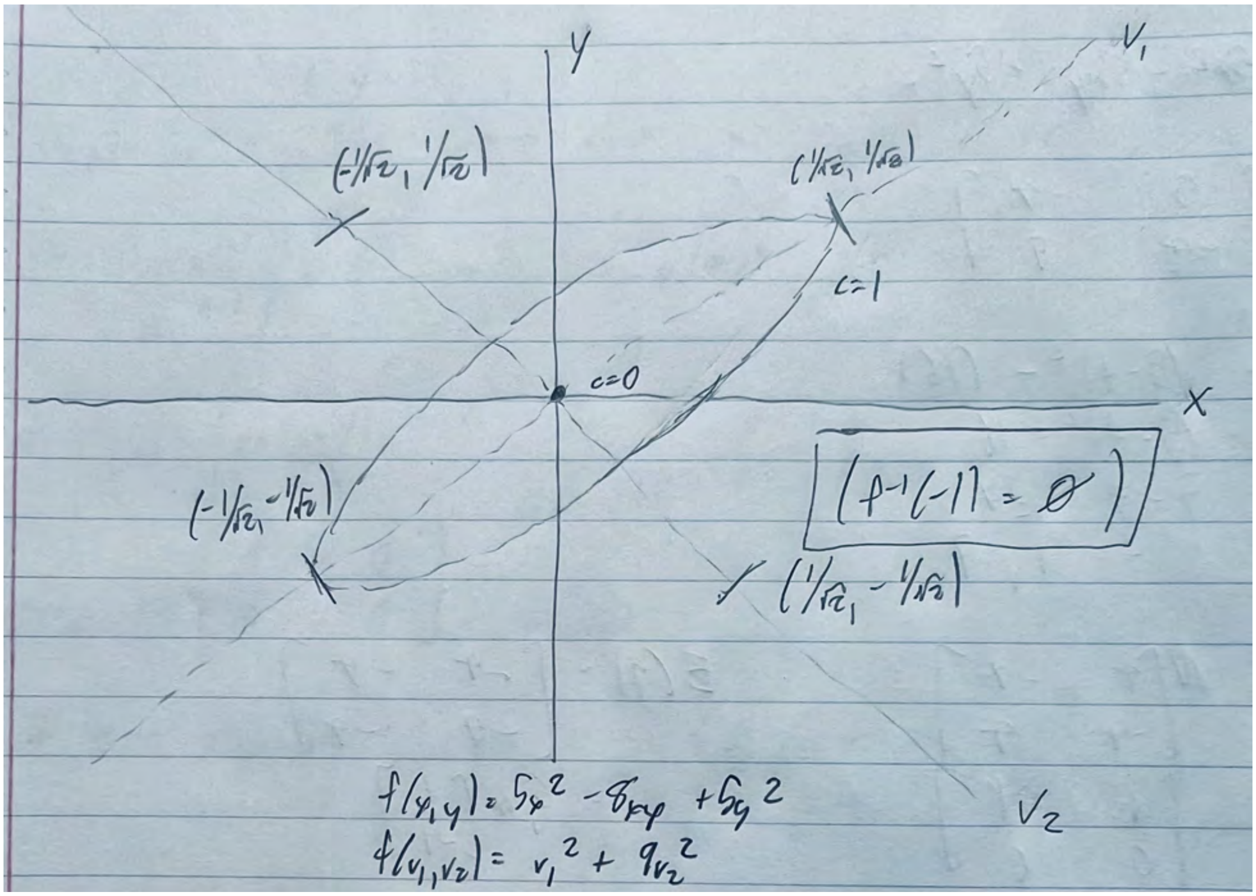
(b) We have $v_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ with $\lambda_1 = 1$ and $v_2 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ with $\lambda_2 = 9$.

(c) Let

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$$

Problem 7 continued on next page. . .

$$\begin{aligned} f(x,y) &= x^T A x \\ &= x^T P \Lambda P^T x \\ &= v^T \Lambda v \\ &= v_1^2 + 9v_2^2 \end{aligned}$$

(d) 

(e) g must have a local minimum at x_0 since $\mathbf{H}g(x_0)$ is positive-definite as $\det(\mathbf{H}g(x_0)) = 9 > 0$ and $a = 5 > 0$.

MATH 222 — Homework #7

Due March 9, 2023

Maxwell Lin

Problem 1

Section 3.3, #28

Find the point on the plane $2x - y + 2z = 20$ nearest the origin.

Solution

Each point on the plane is of the form $(x, y, -x + \frac{y}{2} + 10)$. The distance from this point to the origin is

$$d(x,y) = \sqrt{x^2 + y^2 + \left(-x + \frac{y}{2} + 10\right)^2}.$$

Minimizing this function is equivalent to minimizing the function

$$d^*(x,y) = d(x,y)^2 = x^2 + y^2 + \left(-x + \frac{y}{2} + 10\right)^2$$

since $d(x,y)^2 \geq d(x_0,y_0)^2 \iff d(x,y) \geq d(x_0,y_0)$. The critical points of d^* occur when

$$\nabla d^*(x,y) = \begin{bmatrix} 4x - 20 - y \\ \frac{10}{4}y + 10 - x \end{bmatrix} = 0$$

which only occurs at the point $\left(\frac{40}{9}, \frac{-20}{9}\right)$. The Hessian at this point is

$$\mathbf{H}d^*\left(\frac{40}{9}, \frac{-20}{9}\right) = \begin{bmatrix} 4 & -1 \\ -1 & \frac{10}{4} \end{bmatrix}.$$

Since $\det \mathbf{H}d^*(\frac{40}{9}, \frac{-20}{9}) > 0$ and $a > 0$, we confirm that $\left(\frac{40}{9}, \frac{-20}{9}\right)$ is a local minimum and that $\left(\frac{40}{9}, \frac{-20}{9}, \frac{40}{9}\right)$ is the point on the plane nearest the origin.

Problem 2

Section 3.3, #29

Show that a rectangular box of given volume has minimum surface area when the box is a cube.

Solution

Suppose the rectangular box has a positive volume of $c = xyz$. The surface area is

$$A(x,y,z) = 2xy + 2yz + 2xz.$$

Letting $z = \frac{c}{xy}$, we equivalently obtain

$$A(x,y) = 2xy + \frac{2c}{x} + \frac{2c}{y}.$$

The critical points of A occur when

$$\nabla A(x,y) = \begin{bmatrix} 2y - \frac{2c}{x^2} \\ 2x - \frac{2c}{y^2} \end{bmatrix} = 0.$$

From $\frac{\partial A}{\partial x} = 0$, we obtain $y = \frac{c}{x^2}$. This implies $y^2 = \frac{c^2}{x^4}$. By substituting into $\frac{\partial A}{\partial y} = 0$, we obtain

$$\begin{aligned} 2x - 2c\left(\frac{x^4}{c^2}\right) &= 0 \\ 2x - \frac{2x^4}{c} &= 0 \\ 2x\left(1 - \frac{x^3}{c}\right) &= 0 \end{aligned}$$

which implies that either

$$x = 0 \text{ or } x = c^{1/3}.$$

Since $c = xyz > 0$ we must have that $x = c^{1/3}$. Substituting this into $\frac{\partial A}{\partial x} = 0$, we obtain

$$\begin{aligned} 2y - \frac{2c}{(c^{1/3})^2} &= 0 \\ y &= c^{1/3}. \end{aligned}$$

Lastly, substituting $x = y = c^{1/3}$ into $xyz = c$ we obtain

$$\begin{aligned} (c^{1/3})(c^{1/3})z &= c \\ z &= c^{1/3}. \end{aligned}$$

Therefore, the only critical point occurs when $x = y = z = c^{1/3}$. By computing the Hessian of A at $(c^{1/3}, c^{1/3})$ we obtain

$$\begin{aligned} \mathbf{H}A(c^{1/3}, c^{1/3}) &= \left.\begin{bmatrix} \frac{4c}{x^3} & 2 \\ 2 & \frac{4c}{y^3} \end{bmatrix}\right|_{(c^{1/3}, c^{1/3})} \\ &= \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}. \end{aligned}$$

Since $\det(\mathbf{H}A(c^{1/3}, c^{1/3})) > 0$ and $a > 0$, this critical point is a relative minimum of A. Thus, surface area is minimized when $x = y = z$, that is, when the box is a cube.

Problem 3

Section 3.3, #34

Let $f(x,y) = 5ye^x - e^{5x} - y^5$.
(a) Show that f has a unique critical point and that this point is a local maximum for f.
(b) Show that f is unbounded on the y axis, and thus has no global maximum. [Note that for a function $g(x)$ of a single variable, a unique critical point which is a local extremum is necessarily a global extremum. This example shows that this is not the case for functions of several variables.]

Solution

Problem 3 continued on next page...

(a) The critical points of f occur when

$$\nabla f(x,y) = \begin{bmatrix} 5ye^x - 5e^{5x} \\ 5e^x - 5y^4 \end{bmatrix} = 0.$$

We can rewrite $\frac{\partial f}{\partial x} = 0$ as $y = e^{4x}$. Substituting this into $\frac{\partial f}{\partial y} = 0$, we obtain

$$\begin{aligned} 5e^x - 5(e^{4x})^4 &= 0 \\ 5e^x - 5e^{16x} &= 0 \\ e^x &= e^{16x} \\ x &= 16x \\ x &= 0. \end{aligned}$$

Thus, the only critical point occurs at $(0,1)$. The Hessian of f at $(0,1)$ is

$$\begin{aligned} \mathbf{H}f(0,1) &= \begin{bmatrix} 5ye^x - 25e^{5x} & 5e^x \\ 5e^x & -20y^3 \end{bmatrix}\Bigg|_{(0,1)} \\ &= \begin{bmatrix} -20 & 5 \\ 5 & -20 \end{bmatrix}. \end{aligned}$$

Since $\det(\mathbf{H}f(0,1)) > 0$ and $a < 0$, $(0,1)$ is a local maximum for f.

(b) Along the y-axis, $x = 0$ and we have

$$f(0,y) = 5y - 1 - y^5.$$

We have that

$$\lim_{y\to\infty} f(0,y) = -\infty \quad \text{and} \quad \lim_{y\to-\infty} f(0,y) = \infty.$$

Therefore, f is unbounded on the y-axis, and thus has no global maximum.

Problem 4

Section 3.4, #6.

Find the extrema of f subject to the stated constraints.
$f(x,y,z) = x + y + z$, subject to $x^2 - y^2 = 1, 2x + z = 1$

Solution

The constraint is

$$g(x,y) = \begin{bmatrix} x^2 - y^2 - 1 \\ 2x + z - 1 \end{bmatrix} = 0.$$

We must find all local extrema of $f|_S$ where $S = g^{-1}(0)$. If $f|_s$ has local extrema at (x,y), then there exist x, y, λ_1, and λ_2 so that

$$\begin{aligned} \nabla f(x,y) &= \lambda_1 \nabla g_1(x,y) + \lambda_2 \nabla g_2(x,y), \\ g_1(x,y) &= 0, \\ g_2(x,y) &= 0. \end{aligned}$$

Problem 4 continued on next page...

Computing the gradients and equating components, we obtain

$$1 = \lambda_1 2x + \lambda_2 2 \tag{1}$$
$$1 = \lambda_1(-2y) \tag{2}$$
$$1 = \lambda_2 \tag{3}$$
$$x^2 - y^2 = 1 \tag{4}$$
$$2x + z = 1. \tag{5}$$

Substituting equation (3) into equation (1) and rewriting equation (2) we obtain,

$$-1 = \lambda_1 2x \tag{6}$$
$$-1 = \lambda_1 2y. \tag{7}$$

Thus, equating equations (1) and (2), we obtain

$$\lambda_1 2x = \lambda_1 2y. \tag{8}$$

Suppose $\lambda_1 \neq 0$. Then we can divide equation (8) by $2\lambda_1$ and obtain $x = y$. However, substituting $x = y$ into equation (4) results in $x^2 - x^2 = 1$, a contradiction. Now suppose $\lambda_1 = 0$. Then, substituting into equations (6) and (7), we obtain $-1 = 0$, a contradiction. Therefore, there is no λ_1 for which this set of equations is consistent and therefore, there do not exist any extrema of f subject to the stated constraints.

Problem 5

Section 3.4, #10. (You can do this either by parametrizing S or by thinking of it as a level set of some function on $\mathbb{R}^2$.)

Find the relative extrema of $f \mid S$.
$f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x^2 - y^2, S = \{(x, \cos x) \mid x \in \mathbb{R}\}$.

Solution

Let $c(t) = (t, \cos t)$. Then

$$f(c(t)) = t^2 - \cos^2(t).$$

The relative extrema occur when

$$\begin{aligned} \mathbf{D}f(c(t)) = 2t + 2(\cos t)(\sin t) = 0 \\ 2t + \sin(2t) = 0 \\ -2t = \sin(2t) \\ t = 0. \end{aligned}$$

Applying the second derivative test at $t = 0$ results in

$$\begin{aligned} \mathbf{DD}f(c(0)) &= 2 + 2\cos(2t)\big|_0 \\ &= 4 > 0. \end{aligned}$$

Therefore, the only extrema of $f|_S$ is $(0, 1)$ which is a relative minimum.

Problem 6

Section 3.4, #24

Find the absolute maximum and minimum for the function $f(x, y, z) = x + yz$ on the ball $B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$

Solution

We know the global maximum and minimum exist for $f|_B$ since f is continuous and B is compact. First we locate all critical points of f in the open set $U = \{(x, y, z) \mid x^2 + y^2 + z^2 < 1\}$. Critical points occur when

$$\nabla f = \begin{bmatrix} 1 \\ z \\ y \end{bmatrix} = 0$$

which never occurs so there are no critical points in U.

Now we locate all critical points of f in the boundary $\partial U = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$.
By the Lagrange multiplier theorem, we obtain the set of equations

$$1 = \lambda 2x \tag{1}$$
$$z = \lambda 2y \tag{2}$$
$$y = \lambda 2z \tag{3}$$
$$x^2 + y^2 + z^2 = 1. \tag{4}$$

Substituting equation (3) in equation (4), we obtain

$$\begin{aligned} z &= \lambda 2(\lambda 2z) \\ z &= 4\lambda^2 z \\ \lambda &= \pm\frac{1}{2} \quad \text{assuming } z \neq 0. \end{aligned}$$

Substituting into equation (1), we obtain $x = \pm 1$. Substituting into equation (4), we obtain $y = z = 0$. (If instead $z = 0$, we would receive the same result.) Evaluating f at these critical points, we obtain

$$\begin{aligned} f(1, 0, 0) &= 1 \\ f(-1, 0, 0) &= -1. \end{aligned}$$

Therefore 1 is the absolute maximum occurring at $(1, 0, 0)$ and -1 is the absolute minimum occurring at $(-1, 0, 0)$.

Problem 7

Suppose $C_1, C_2, C_3, \dots \subset \mathbb{R}^n$ are any collection of closed sets. (This could be a collection of finitely many or infinitely many sets.) Show that the intersection $\bigcap_i C_i$ is closed. Likewise, show that if $C_1, C_2, C_3, \dots$ are compact sets, then $\bigcap_i C_i$ is compact. (Hint: use a corresponding statement about open sets proven in HW 2.)

Solution

First we prove that $(\bigcap_i C_i)' = \bigcup_i C_i'$.

Problem 7 continued on next page...

Proof 1.

$$x \in \left(\bigcap_i C_i\right)' \iff \neg(x \in C_i \ \forall i)$$
$$\iff x \notin C_i \ \exists i$$
$$\iff x \in C_i' \ \exists i$$
$$\iff x \in \bigcup_i C_i'.$$

Since each set is contained within the other, $(\bigcap_i C_i)' = \bigcup_i C_i'$. □

Proof 2. From HW 2, we know that the infinite union of open sets is open. Thus, $\bigcup_i C_i'$ is open since the complement of an closed set is open. Therefore, $(\bigcap_i C_i)' = \bigcup_i C_i'$ is open. Taking the complement, we get that $\bigcap_i C_i$ is closed as required. □

Proof 3. We have already shown that $C = \bigcap_i C_i$ is closed. Therefore, to show that C is compact, we must prove that C is bounded. Suppose $x \in C$. We need to show that there exists $R > 0$ such that $\|x\| < R$. Since $C = \bigcap_i C_i \subset C_i$ for $\forall i$, we also have that $x \in C_i$. Since C_i is bounded (since it is compact), there exists some R_i such that $\|x\| < R_i$. Therefore, C is both closed and bounded, and therefore compact. □

Problem 8

Consider Example 4 on page 190. Let

$$S = \{(x, y, z) \mid xy + yz + xz = 5, x \geq 0, y \geq 0, z \geq 0\}$$

and let $f(x, y, z) = xyz$. As outlined in class, we will prove carefully that the function f actually achieves a global maximum on S, which must then be the value that is found in the example.
(a) Show that if $(x, y, z) \in S$, and x, y, z are all nonzero, then

$$f(x, y, z) < \min\{25/x, 25/y, 25/z\}.$$

(b) Let

$$S' = \{(x, y, z) \in S \mid x \leq 25, y \leq 25, z \leq 25\}$$

Show that S' is compact, and hence that $f|_{S'}$ attains a maximum on S'. (Hint: Use Problem 7.) Show that this maximum is at least 2.
(c) Show that if (x, y, z) is a point of S not in S', then $f(x, y, z) < 1$.
(d) Deduce that f has a global maximum on S.

Solution

(a) We must show that xyz is less than $25/x$, $25/y$, and $25/z$. We have that

$$\begin{aligned}
(xy - xz)^2 = x^2y^2 - 2x^2yz + x^2z^2 &\geq 0 \\
(xy + xz)^2 = x^2y^2 + 2x^2yz + x^2z^2 &\geq 4x^2yz && \text{add } 4x^2yz \text{ to both sides} \\
25 = (xy + yz + xz)^2 > (xy + xz)^2 &\geq 4x^2yz && \text{since } yz \text{ positive} \\
\frac{25}{x} > \frac{25}{4x} &> xyz && \text{since x positive.}
\end{aligned}$$

Since f and S are symmetric with regards to x, y, and z, the same reasoning will show that xyz is less than $25/y$ and $25/z$. Therefore, $f(x, y, z) < \min\{25/x, 25/y, 25/z\}$.

Problem 8 continued on next page. . .

(b) S' is bounded since $\|(x,y,z)\| \le \sqrt{(25^2)+(25^2)+(25^2)}$ for all $(x,y,z) \in S'$.

Consider

$$\begin{aligned} C_1 &= \left\{(x,y,z) \in \mathbb{R}^3 \mid 0 \le x \le 25\right\} \\ C_2 &= \left\{(x,y,z) \in \mathbb{R}^3 \mid 0 \le y \le 25\right\} \\ C_3 &= \left\{(x,y,z) \in \mathbb{R}^3 \mid 0 \le z \le 25\right\} \\ T &= \left\{(x,y,z) \in \mathbb{R}^3 \mid xy+yz+xz = 5\right\}. \end{aligned}$$

Note that $S' = C_1 \cap C_2 \cap C_3 \cap T$. We have that T is a closed set (since it is a level set of a continuous function) and that C_i is a closed set since it includes the boundary. Thus, by Problem 7, S' is closed since the intersection of closed sets is closed. Since, S' is bounded and closed, it is compact. Since S' is compact and f is continuous, $f|_{S'}$ attains a maximum on S'.

By the Lagrange multiplier theorem, we obtain the set of equations

$$\begin{aligned} yz &= \lambda(y+z) && (1) \\ xz &= \lambda(x+z) && (2) \\ xy &= \lambda(x+y) && (3) \\ xy+yz+xz &= 5. && (4) \end{aligned}$$

We know $x \neq 0$ since if $x = 0$, then $yz = 5$ (equation 4) and $0 = \lambda z$ (equation 2). This implies that $\lambda = 0$ since $z \neq 0$. But, this leads to a contradiction since $yz = 0$ (equation 1). By the same reasoning, $y \neq 0$ and $z \neq 0$.

Elimination of λ in equations (1) and (2) as well as (2) and (3) results in

$$\begin{aligned} \frac{yz}{y+z} &= \frac{xz}{x+z} \implies x = y && (5) \\ \frac{xz}{x+z} &= \frac{xy}{x+y} \implies y = z. && (6) \end{aligned}$$

Therefore, from equation (4) we obtain that $x = y = z = \sqrt{\frac{5}{3}}$. Since $\left(\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right) \in S'$, we have that the maximum of f on S' is $\left(\frac{5}{3}\right)^{3/2} \approx 2.151 > 2$ as required.

(c) If $(x,y,z) \in S \setminus S'$, then $x > 25$, $y > 25$, or $z > 25$. Let us assume $x > 25$. From part (a), we know that $f(x,y,z) < \min\{25/x, 25/y, 25/z\} \le 25/x < 1$. By the same reasoning, $f(x,y,z) < 1$ if $y > 25$ or $z > 25$ instead.

(d) By part (b), there is some $\mathbf{x}_0 \in S'$ such that $f(\mathbf{x}_0) \ge 2$ and for every $\mathbf{a} \in S'$, $f(\mathbf{a}) \le f(\mathbf{x}_0)$. By part (c), for every $\mathbf{b} \in S \setminus S'$, we have $f(\mathbf{b}) < 1 < 2 \le f(\mathbf{x}_0)$. Combining these statements shows that $f(\mathbf{x}) \le f(\mathbf{x}_0)$ for every $\mathbf{x} \in S$, and hence f has a global maximum at $\mathbf{x}_0$.

MATH 222 — Homework #8

Due March 23, 2023

Maxwell Lin

Problem 1

Section 5.2, #4

Evaluate over the region R:

$$\iint_R \frac{y}{1+x^2}\,dx\,dy, \quad R\colon [0,1]\times[-2,2]$$

Solution

$$\begin{aligned}
\int_{-2}^{2}\int_{0}^{1}\frac{y}{1+x^2}\,dx\,dy &= \int_{-2}^{2} y\left[\arctan x\right]_{x=0}^{1}\,dy \\
&= \frac{\pi}{4}\int_{-2}^{2} y\,dy \\
&= \frac{\pi}{4}\left[\frac{y^2}{2}\right]_{y=-2}^{2} \\
&= \frac{\pi}{4}(0) \\
&= 0.
\end{aligned}$$

Problem 2

Section 5.2, #9

Let f be continuous on $[a,b]$ and g continuous on $[c,d]$. Show that

$$\iint_R [f(x)g(y)]\,dx\,dy = \left[\int_a^b f(x)\,dx\right]\left[\int_c^d g(y)\,dy\right]$$

where $R=[a,b]\times[c,d]$.

Solution

Proof. Since f is continuous on $[a,b]$ and g is continuous on $[c,d]$, $f(x)g(y)$ is continuous on R. By Fubini's Theorem,

$$\iint_R [f(x)g(y)]\,dx\,dy = \int_c^d \left[\int_a^b f(x)g(y)\,dx\right]dy.$$

When we evaluate this integral, we first hold y fixed and integrate with respect to x. Since y is fixed, $g(y)$ is a constant. Therefore, by the homogeneity of integrals,

$$\int_c^d \left[\int_a^b f(x)g(y)\,dx\right]dy = \int_c^d g(y)\left[\int_a^b f(x)\,dx\right]dy.$$

Likewise, since the inner integral does not contain any y, it is constant and we obtain

$$\int_c^d g(y)\left[\int_a^b f(x)\,dx\right]dy = \left[\int_a^b f(x)\,dx\right]\left[\int_c^d g(y)\,dy\right].$$

$\square$

Problem 3

Section 5.3, #4(a, c, d)

Evaluate the following integrals and sketch the corresponding regions.

(a) $\int_{-3}^{2}\int_0^{y^2}\left(x^2+y\right)\,dx\,dy$

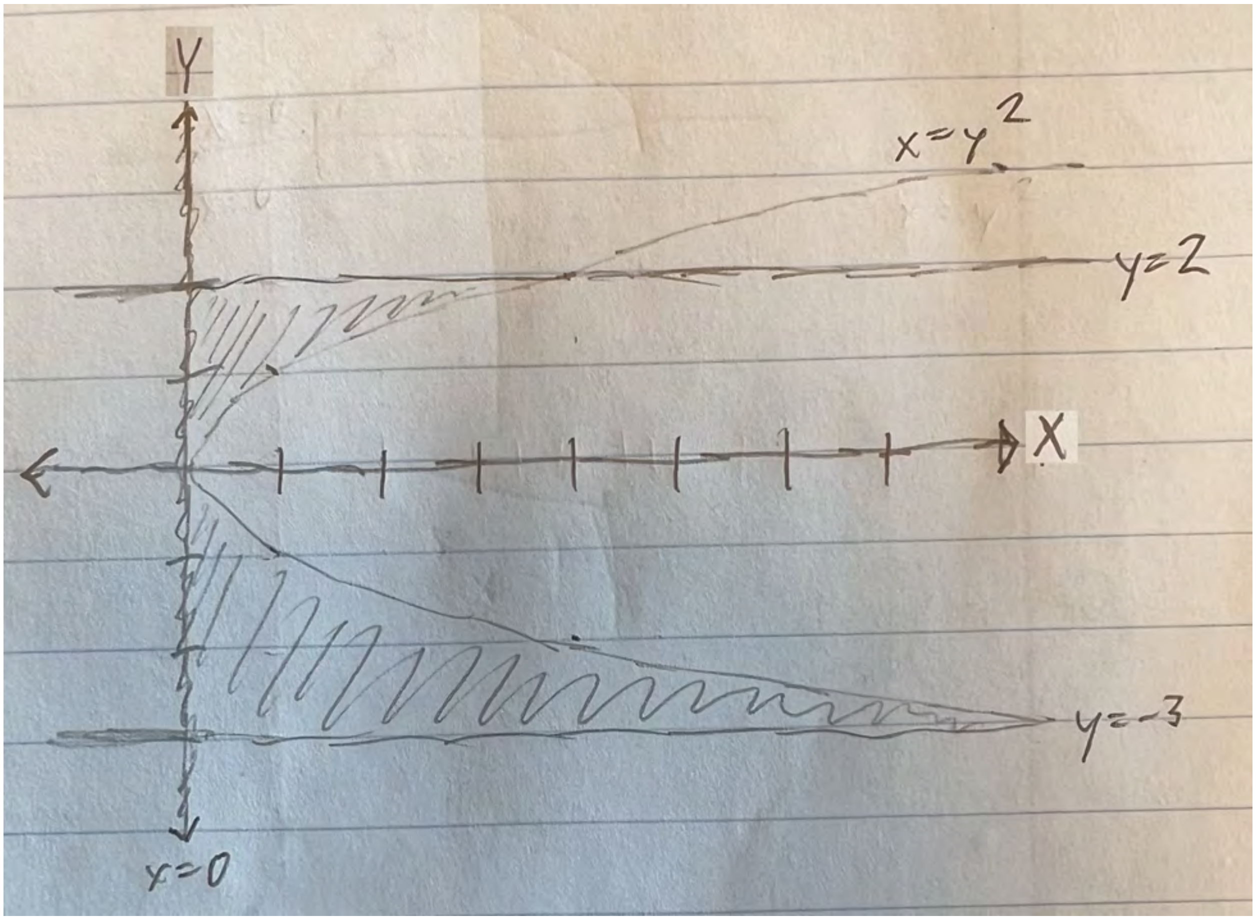
(c) $\int_0^1\int_0^{\left(1-x^2\right)^{1/2}}\,dy\,dx$

(d) $\int_0^{\pi/2}\int_0^{\cos x} y\sin x\,dy\,dx$

Solution

(a) We have

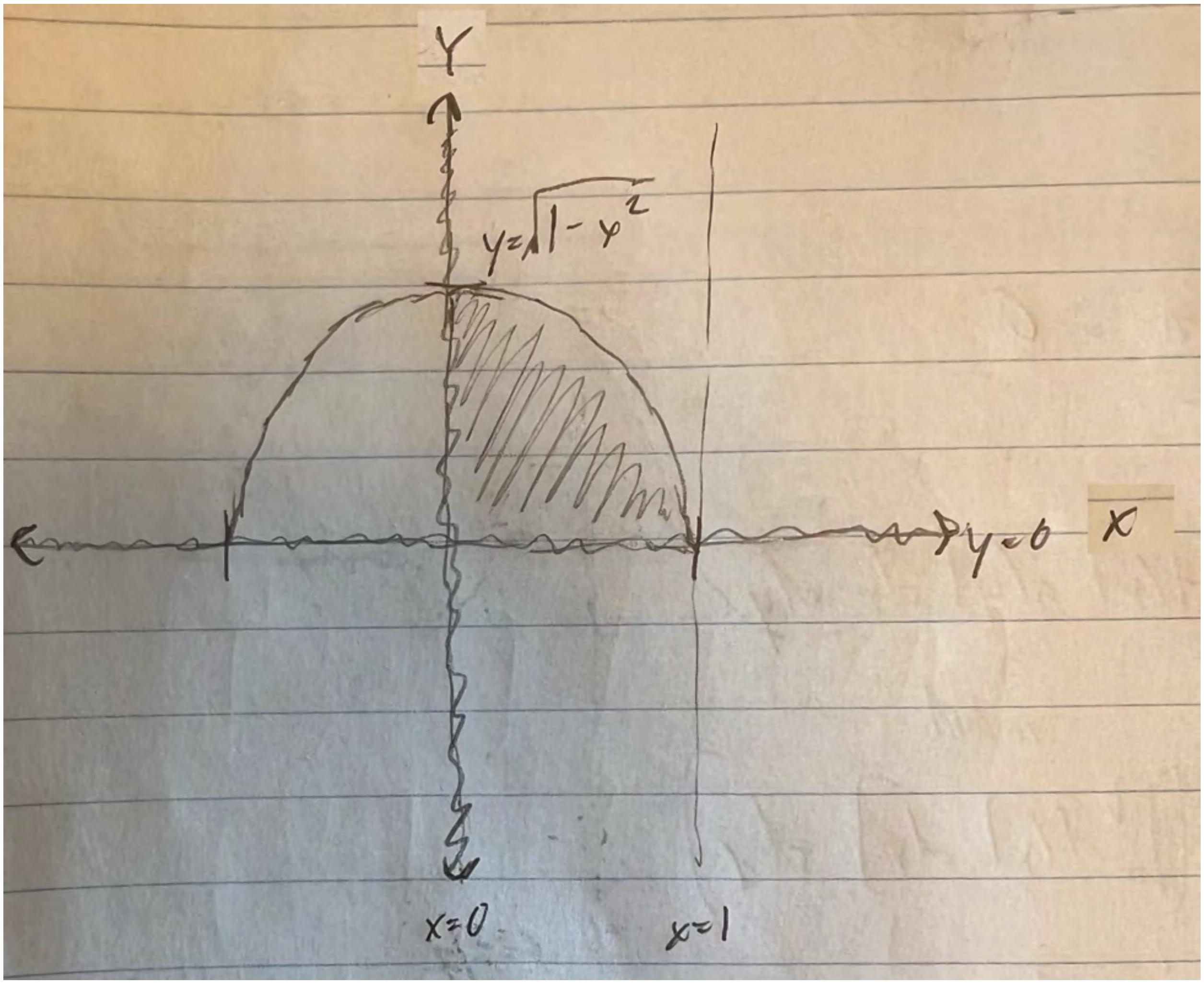
$$\begin{aligned}\int_{-3}^{2}\int_0^{y^2}\left(x^2+y\right)\,dx\,dy &= \int_{-3}^{2}\left[\frac{x^3}{3}+yx\right]_{x=0}^{y^2}\,dy\\ &= \int_{-3}^{2}\frac{y^6}{3}+y^3\,dy\\ &= \left[\frac{y^7}{21}+\frac{y^4}{4}\right]_{y=-3}^{2}\\ &= \frac{7895}{84}.\end{aligned}$$



(c) We have

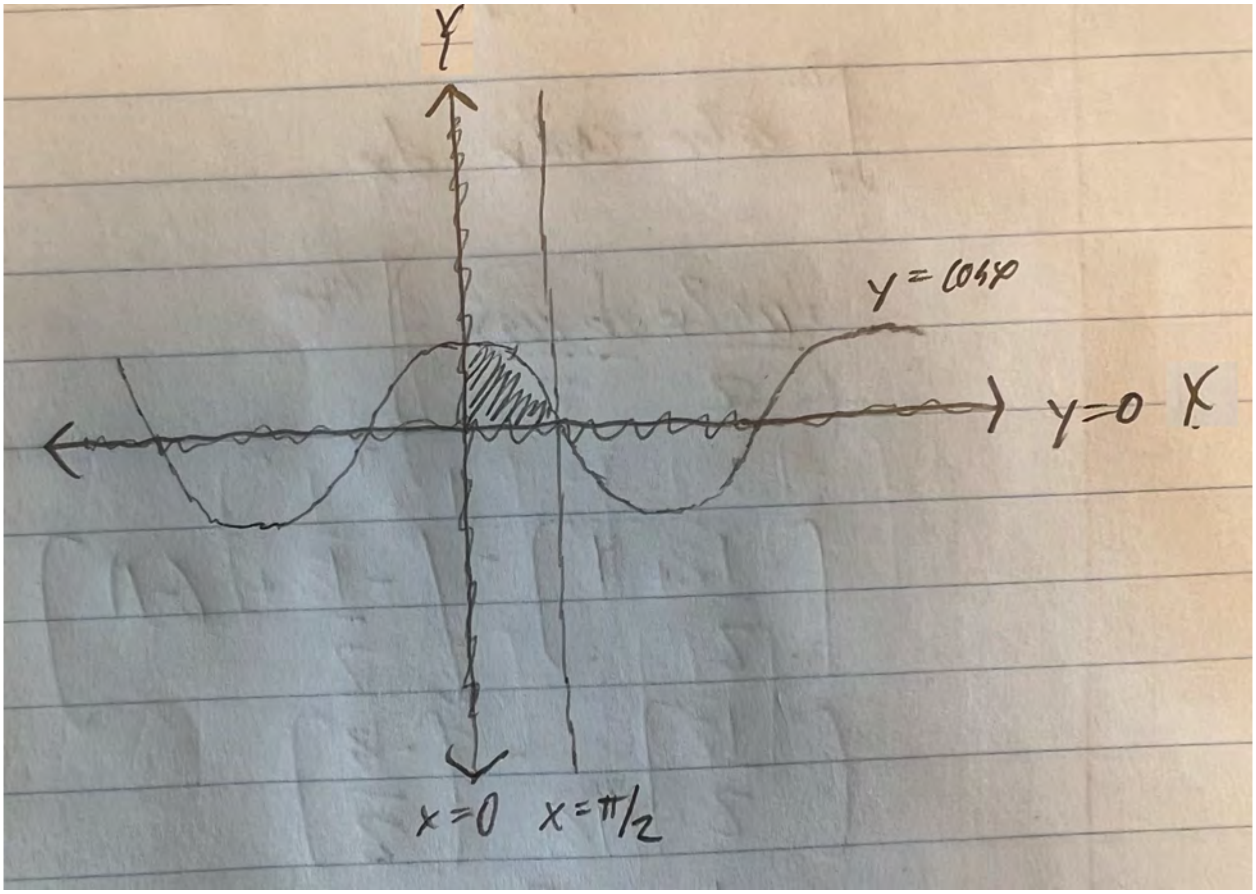
$$\begin{aligned}
\int_0^1 \int_0^{\left(1-x^2\right)^{1/2}} dy\, dx &= \int_0^1 (1-x^2)^{1/2}\, dx \\
&= \int_{\pi/2}^0 (1-\cos^2\theta)^{1/2}(-\sin\theta)\, d\theta \qquad \text{substituting } x=\cos\theta \\
&= -\int_{\pi/2}^0 \sin^2\theta\, d\theta \qquad \text{using } \sin^2\theta+\cos^2\theta=1 \\
&= -\int_{\pi/2}^0 \frac{1}{2}-\frac{\cos(2\theta)}{2}\, d\theta \\
&= -\left[\frac{\theta}{2}-\frac{\sin(2\theta)}{4}\right]_{\theta=\pi/2}^0 \\
&= \frac{\pi}{4}.
\end{aligned}$$

Problem 3 continued on next page. . .



(d) We have

$$
\begin{aligned}
\int_0^{\pi/2}\int_0^{\cos x} y\sin x\,dy\,dx &= \int_0^{\pi/2}\left[\frac{y^2}{2}\sin x\right]_{y=0}^{\cos x} dx \\
&= \int_0^{\pi/2}\frac{(\cos x)^2\sin x}{2}\,dx \\
&= -\frac{1}{2}\int_1^0 u^2\,du \qquad \text{substituting u} = \cos x \\
&= -\frac{1}{2}\left[\frac{u^3}{3}\right]_{u=1}^{0} \\
&= \frac{1}{6}.
\end{aligned}
$$



Problem 4

Section 5.3, #12

Evaluate the following double integral:

$$\iint_D \cos y \, dx \, dy$$

where the region D is bounded by $y = 2x, y = x$, $x = \pi$, and $x = 2\pi$

Solution

We have

$$\begin{aligned}
\iint_D \cos y \, dx \, dy &= \int_{\pi}^{2\pi} \int_{x}^{2x} \cos y \, dy \, dx \\
&= \int_{\pi}^{2\pi} [\sin y]_{y=x}^{2x} \, dx \\
&= \int_{\pi}^{2\pi} \sin(2x) - \sin x \, dx \\
&= \left[\frac{-\cos(2x)}{2} + \cos x \right]_{x=\pi}^{2\pi} \\
&= 2.
\end{aligned}$$

Problem 5

Section 5.4, #4(a, c)
Find

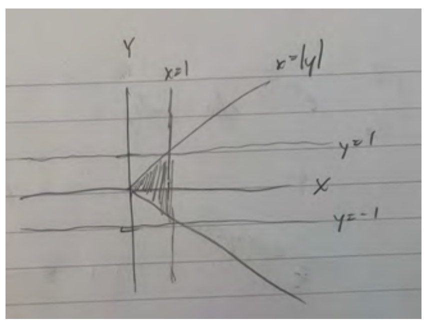
(a) $\int_{-1}^{1}\int_{|y|}^{1}(x+y)^2\,dx\,dy$

(c) $\int_{0}^{4}\int_{y/2}^{2}e^{x^2}\,dx\,dy$

Solution

(a) We have

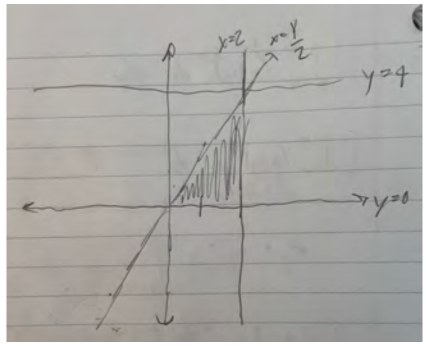
$$\begin{aligned}
\int_{-1}^{1}\int_{|y|}^{1}(x+y)^2\,dx\,dy &= \int_{0}^{1}\int_{-x}^{x}(x+y)^2\,dy\,dx \\
&= \int_{0}^{1}\left[x^2y+xy^2+\frac{y^3}{3}\right]_{y=-x}^{x}dx \\
&= \int_{0}^{1}\frac{8x^3}{3}\,dx \\
&= \left[\frac{2x^4}{3}\right]_{x=0}^{1} \\
&= \frac{2}{3}.
\end{aligned}$$



(c) We have

$$\begin{aligned}
\int_{0}^{4}\int_{y/2}^{2}e^{x^2}\,dx\,dy &= \int_{0}^{2}\int_{0}^{2x}e^{x^2}\,dy\,dx \\
&= \int_{0}^{2}2xe^{x^2}\,dx \\
&= \int_{0}^{4}e^{u}\,du \qquad \text{substituting } u=x^2 \\
&= e^4-1.
\end{aligned}$$

Problem 5 continued on next page. . .



Problem 6

Section 5.5, #15
$\int_0^1 \int_1^2 \int_2^3 \cos[\pi(x+y+z)]\,dx\,dy\,dz$

Solution

We have

$$\begin{aligned}
\int_0^1 \int_1^2 \int_2^3 \cos[\pi(x+y+z)]\,dx\,dy\,dz &= \int_0^1 \int_1^2 \left[\frac{\sin(x\pi+y\pi+z\pi)}{\pi}\right]_{x=2}^3 dy\,dz \\
&= \int_0^1 \int_1^2 \frac{\sin(3\pi+y\pi+z\pi)}{\pi} - \frac{\sin(2\pi+y\pi+z\pi)}{\pi}\,dy\,dz \\
&= \int_0^1 \left[\frac{-\cos(3\pi+y\pi+z\pi)}{\pi^2} + \frac{\cos(2\pi+y\pi+z\pi)}{\pi^2}\right]_{y=1}^2 dz \\
&= \int_0^1 \frac{-\cos(5\pi+z\pi)}{\pi^2} + \frac{2\cos(4\pi+z\pi)}{\pi^2} - \frac{\cos(3\pi+z\pi)}{\pi^2}\,dz \\
&= \left[\frac{-\sin(5\pi+z\pi)}{\pi^3} + \frac{2\sin(4\pi+z\pi)}{\pi^3} - \frac{\sin(3\pi+z\pi)}{\pi^3}\right]_{z=0}^1 \\
&= 0 \qquad \text{since each term is 0.}
\end{aligned}$$

Problem 7

Compute the volume of the 3-dimensional ball of radius R, $B_R = \{(x,y,z) \mid x^2+y^2+z^2 \le R^2\}$, in the following three ways (and check that your answers agree):
(a) Using Cavalieri's principle, as an integral of the area of the cross-sectional circles perpendicular to the x-axis, ranging from $x=-R$ to $x=R$.
(b) As twice the volume of the region below the graph of $z=\sqrt{R^2-x^2-y^2}$ lying above the disk of radius R.
(c) As the value of the constant function 1 integrated over B.

Solution

(a) The volume of B_R is

$$\int_{-R}^{R} \pi r^2\,dx$$

Problem 7 continued on next page. . .

where r is the radius of the cross-sectional circle perpendicular to the x-axis for fixed x. Since $x^2+y^2+z^2 \leq R^2$, we have $y^2+z^2 \leq R^2-x^2$. Thus, $r=\sqrt{R^2-x^2}$.

Substituting we obtain,

$$\begin{aligned}\int_{-R}^{R} \pi(R^2-x^2)\,dx &= \left[\pi R^2 x - \frac{\pi x^3}{3}\right]_{x=-R}^{R} \\ &= \frac{4}{3}\pi R^3.\end{aligned}$$

(b) We have that

$$D_R = \left\{ (x,y) \;\middle|\; \begin{array}{l} -R \leq x \leq R \\ -\sqrt{R^2-x^2} \leq y \leq \sqrt{R^2-x^2} \end{array} \right\}.$$

Thus, the volume of B_R is

$$2\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2-x^2-y^2}\,dy\,dx.$$

First, we compute the inner integral.

$$\begin{aligned}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2-x^2-y^2}\,dy &= \int_{-r}^{r} \sqrt{r^2-y^2}\,dy && \text{Let } r=\sqrt{R^2-x^2}. \\ &= \int_{-\pi/2}^{\pi/2} \sqrt{r^2-r^2\sin^2\theta}(r\cos\theta)\,d\theta && \text{Let } y = r\sin\theta \\ &= \int_{-\pi/2}^{\pi/2} \sqrt{r^2\cos^2\theta}(r\cos\theta)\,d\theta && \text{since } \sin^2\theta+\cos^2\theta=1 \\ &= \int_{-\pi/2}^{\pi/2} |r\cos\theta|(r\cos\theta)\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\,d\theta && \text{since } \cos\theta \geq 0 \text{ on } [-\pi/2,\pi/2] \\ &= r^2\int_{-\pi/2}^{\pi/2} \frac{\cos(2\theta)+1}{2}\,d\theta && \text{since } \cos(2\theta) = 2\cos^2\theta-1 \\ &= \frac{r^2}{2}\left[\frac{\sin(2\theta)}{2}+\theta\right]_{-\pi/2}^{\pi/2} \\ &= \frac{\pi r^2}{2}.\end{aligned}$$

Now, we compute the outer integral.

$$\int_{-R}^{R} \frac{\pi r^2}{2}\,dx = \frac{1}{2}\int_{-R}^{R} \pi r^2\,dx = \frac{1}{2}\left[\frac{4}{3}\pi R^3\right] \qquad \text{Part (a).}$$

Multiplying by 2, we obtain $\frac{4}{3}\pi R^3$ as required.

(c) We have that

$$B_R = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} -R \leq x \leq R \\ -\sqrt{R^2-x^2} \leq y \leq \sqrt{R^2-x^2} \\ -\sqrt{R^2-x^2-y^2} \leq z \leq \sqrt{R^2-x^2-y^2} \end{array} \right\}.$$

Problem 7 continued on next page...

Therefore, the volume of B_R is

$$\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} 1\,dz\,dy\,dx = 2\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2-x^2-y^2}\,dy\,dx$$
$$= \frac{4}{3}\pi R^3 \qquad \text{Part (b).}$$

Problem 8

Viewing your answer to the previous problem as a function of R, compute its derivative with respect to R, and give a geometric interpretation for what this derivative represents. (You may have encountered this formula before.) Do the same thing with the formula for the area of a circle.

Solution

We have

$$\frac{d}{dR}\left[\frac{4}{3}\pi R^3\right] = 4\pi R^2 \qquad \text{Surface area of a sphere}$$
$$\frac{d}{dr}\left[\pi r^2\right] = 2\pi r \qquad \text{Circumference of a circle}$$

The definition of the derivative states

$$f'(R) = \lim_{h\to 0}\frac{f(R+h)-f(R)}{h}.$$

That is

$$f'(R)h \approx f(R+h) - f(R)$$

for small values of h.

Geometrically, we add a thin shell of thickness h around a sphere (or circle) of radius R. Since h is small, this added volume should approximately equal the surface area, $f'(R)$, multiplied by the added thickness h.

MATH 222 — Homework #9

Due April 6, 2023

Maxwell Lin

Problem 1

Section 6.1, #8
Let D^* be the parallelogram bounded by the lines $y = 3x - 4, y = 3x, y = \frac{1}{2}x$, and $y = \frac{1}{2}(x+4)$. Let $D = [0,1] \times [0,1]$. Find a T such that D is the image of D^* under T.

Solution

Since both D^* and D are parallelograms, one such T is a linear transformation that maps the vertices of D^* to D. Thus, we have

$$\begin{aligned} T(0,0) &= (0,0) && (1) \\ T(4/5, 12/5) &= (0,1) && (2) \\ T(12/5, 16/5) &= (1,1) && (3) \\ T(8/5, 4/5) &= (1,0). && (4) \end{aligned}$$

Reading off equations (2) and (4) give us the matrix A^{-1} for T^{-1}

$$A^{-1} = \begin{bmatrix} T^{-1}(e_1) & T^{-1}(e_2) \end{bmatrix} = \begin{bmatrix} 8/5 & 4/5 \\ 4/5 & 12/5 \end{bmatrix}.$$

Inverting A gives us

$$A = \begin{bmatrix} 3/4 & -1/4 \\ -1/4 & 1/2 \end{bmatrix}.$$

Thus, T is given as

$$T(x,y) = \left(\frac{3}{4}x - \frac{1}{4}y, -\frac{1}{4}x + \frac{1}{2}y \right).$$

Problem 2

Section 6.2, #2
Suggest a substitution/transformation that will simplify the following integrands, and find their Jacobians.
(a) $\iint_R (5x+y)^3(x+9y)^4 dA$
(b) $\iint_R x\sin(6x+7y) - 3y\sin(6x+7y) dA$

Solution

(a) A good substitution may be

$$u = 5x + y$$
$$v = x + 9y.$$

That is,

$$T^{-1}(x,y) = (5x+y, x+9y).$$

The Jacobian determinant of T is

$$|\det(\mathbf{D}T)| = \left|\frac{1}{\det(\mathbf{D}T^{-1})}\right| = \frac{1}{44}.$$

(b) The integrand can be rewritten as

$$x\sin(6x+7y) - 3y\sin(6x+7y) = \sin(6x+7y)(x-3y)$$

Thus, a good substitution may be

$$\begin{aligned} u &= 6x+7y \\ v &= x-3y. \end{aligned}$$

That is,

$$T^{-1}(x,y) = (6x+7y, x-3y).$$

The Jacobian determinant of T is

$$|\det(\mathbf{D}T)| = \left|\frac{1}{\det(\mathbf{D}T^{-1})}\right| = \frac{1}{25}.$$

Problem 3

Let D be the region $0 \le y \le x$ and $0 \le x \le 1$. Evaluate

$$\iint_D (x+y)\,dx\,dy$$

by making the change of variables $x = u+v$, $y = u-v$. Check your answer by evaluating the integral directly by using an iterated integral.

Solution

The transformation T is

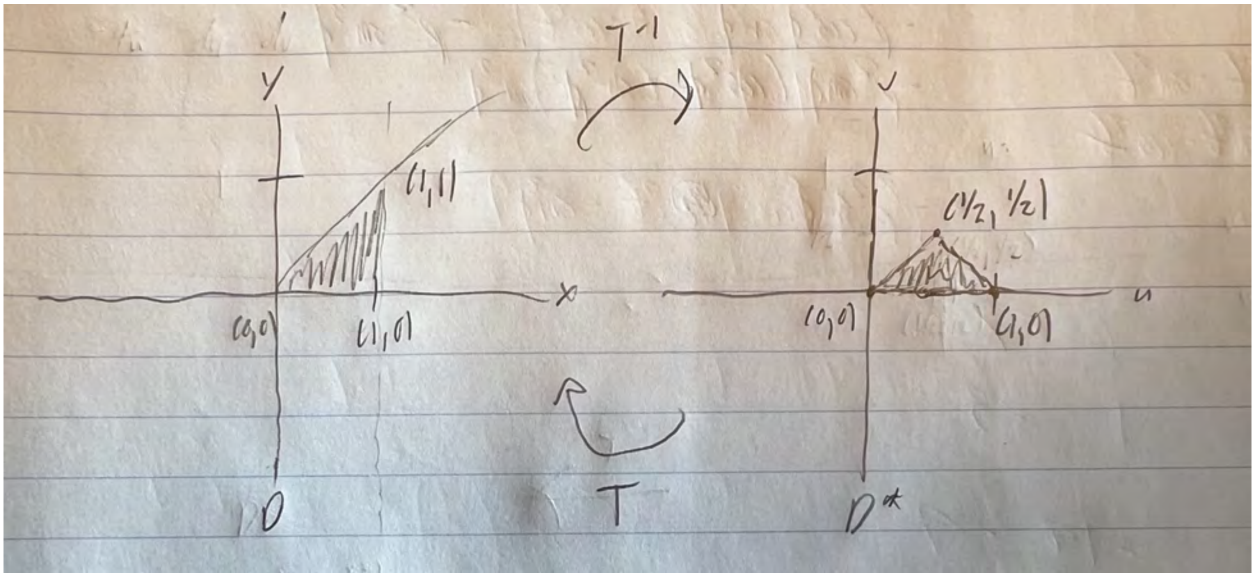
$$T(u,v) = (u+v, u-v)$$

which has the Jacobian determinant 2. Since T is bijective, we can solve for T^{-1}

$$T^{-1}(x,y) = \left(\frac{1}{2}x + \frac{1}{2}y, \frac{1}{2}x - \frac{1}{2}y\right)$$

Since T is a linear transformation, it maps lines to lines. Thus, using T^{-1} we observe that T maps the triangle with vertices $(0,0)$, $(1,0)$ and $(1/2,1/2)$ to D.

Problem 3 continued on next page. . .



Therefore, we have

$$\begin{aligned}\iint_D (x+y)\,dx\,dy &= \int_0^{1/2}\int_v^{1-v} 4u\,du\,dv \\ &= \int_0^{1/2} \left[2u^2\right]_{u=v}^{u=1-v}\,dv \\ &= \int_0^{1/2} 2(1-2v)\,dv \\ &= 2[v-v^2]_0^{1/2} \\ &= \frac{1}{2}.\end{aligned}$$

Now we evaluate the integral directly.

$$\begin{aligned}\iint_D (x+y)\,dx\,dy &= \int_0^1\int_0^x x+y\,dy\,dx \\ &= \int_0^1 \left[xy+\frac{y^2}{2}\right]_{y=0}^{y=x}\,dx \\ &= \int_0^1 \frac{3x^2}{2}\,dx \\ &= \left[\frac{x^3}{2}\right]_0^1 \\ &= \frac{1}{2}\end{aligned}$$

as required.

Problem 4

Section 6.2, #8

Problem 4 continued on next page...

Define $T(u,v) = \left(u^2 - v^2, 2uv\right)$. Let D^* be the set of (u,v) with $u^2+v^2 \leq 1, u \geq 0, v \geq 0$. Find $T\left(D^*\right) = D$. Evaluate $\int\int_D \, dx\, dy$.

Solution

We can transform D^* into polar coordinates with the transformation $(u,v) = P(r,\theta) = (r\cos\theta, r\sin\theta)$. We obtain

$$u^2 + v^2 = r^2 \leq 1$$
$$u = r\cos\theta \geq 0 \qquad v = r\sin\theta \geq 0$$

and thus,

$$D' = \left\{ (r,\theta) \,\middle|\, \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq \dfrac{\pi}{2} \end{array} \right\}.$$

Therefore, to determine $T(D^*)$ we can equivalently solve for $T(P(D'))$ where D' is D^* expressed in polar coordinates. We have

$$\begin{aligned}(T \circ P) &= ((r\cos\theta)^2 - (r\sin\theta)^2, 2r^2\cos\theta\sin\theta) \\ &= r^2(\cos(2\theta), \sin(2\theta)).\end{aligned}$$

That is, $T(D^*) = (T \circ P)(D')$ is the closed region of radius 1 from angle 0 to π—the top half of the unit circle.

$$D = \left\{ (x,y) \,\middle|\, \begin{array}{l} -1 \leq x \leq 1 \\ 0 \leq y \leq \sqrt{1-x^2} \end{array} \right\}$$

With this geometric picture, $\iint_D \, dx\, dy$ must be $\pi/2$. We evaluate

$$\det(\mathbf{D}T) = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix} = 4(u^2+v^2).$$

Since $\det(\mathbf{D}T) = 0$ only at $(0,0)$ which is on the boundary of D^*, we can change variables as follows

$$\begin{aligned}\iint_D \, dx\, dy &= \iint_{D^*} 4(u^2+v^2)\, du\, dv \\ &= \int_0^{\pi/2}\int_0^1 4r^2(r)\, dr\, d\theta \\ &= \frac{\pi}{2}\end{aligned}$$

as required.

Problem 5

In this problem, we will compute the formula for the n-dimensional volume of an n-dimensional ball of radius R,

$$D_R^n = \left\{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq R\right\}.$$

Let $V_n(R)$ be the volume of this ball, i.e.

$$V_n(R) = \int_{D_R^n} 1 dV$$

Write D^n for D_1^n, and let $\alpha_n = V_n(1)$.
(a) What are α_1, α_2, and α_3?
(b) Prove that for any $R > 0$, we have

$$V_n(R) = \alpha_n R^n$$

(Hint: Use the function $T : D_1^n \to D_R^n$ given by $T(\mathbf{x}) = R\mathbf{x}$ and the change-of-coordinates formula.)
(c) Prove that for any $n \geq 3$, we have

$$V_n(R) = \int_{D_R^2} V_{n-2}\left(\sqrt{R^2 - x^2 - y^2}\right) dA.$$

By computing this integral using polar coordinates and the formula from part (b), show that

$$V_n(R) = \frac{2\pi R^2}{n} V_{n-2}(R).$$

(d) When n is even, show that

$$\alpha_n = \frac{\pi^{n/2}}{(n/2)!},$$

and hence

$$V_n(R) = \frac{\pi^{n/2}}{(n/2)!} R^n.$$

Note: There is a smooth function called Γ, known as Euler's Gamma function, which is defined for all non-negative real numbers and "connects the dots" of the factorial function: when n is a positive integer, it satisfies $\Gamma(n) = (n-1)!$, and for all real numbers, it has $\Gamma(x+1) = x\Gamma(x)$. The above formula then generalizes to

$$V_n(R) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} R^n$$

which now makes sense for both even and odd n. To learn more about Γ, you should take Math 333 (Complex Analysis).
(e) Deduce that the limit as $n \to \infty$ of α_n, taken over all even n, is 0 .
(f) In Homework 1, we defined the n-dimensional cube

$$C^n = \{\mathbf{x} \in \mathbb{R}^n \| x_i \mid \leq 1 \text{ for all } i = 1, \ldots, n\}$$

in which D^n is inscribed. (That is, we saw that $D^n \subset C^n$, but that $D_R^n \not\subset C^n$ for $R > 1$.) What is the volume of C^n ? How does this behave as $n \to \infty$?

Solution

(a) We have

$$\begin{aligned} \alpha_1 = V_1(1) &= \int_{D_1^1} 1 \, dv \\ &= \int_{-1}^{1} 1 \, dx \\ &= 2 \end{aligned}$$

$$\begin{aligned}\alpha_2 = V_2(1) &= \int_{D_1^2} 1\,dv \\ &= \int_0^{2\pi}\int_0^1 r\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\,d\theta \\ &= \pi\end{aligned}$$

$$\begin{aligned}\alpha_3 = V_3(1) &= \int_{D_1^3} 1\,dv \\ &= \frac{4\pi}{3} \qquad \text{HW 8 Problem 7.}\end{aligned}$$

(b) The transformation T is

$$T(x_1,\dots,x_n) = (Rx_1,\dots,Rx_n).$$

The Jacobian is

$$\mathbf{D}T = \begin{bmatrix} R & 0 & 0 & \cdots & 0 \\ 0 & R & 0 & \cdots & 0 \\ 0 & 0 & R & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & R \end{bmatrix} = RI.$$

Since this matrix can be obtained by multiplying each row of the identity matrix by R, we have that

$$\det(\mathbf{D}T) = R^n\det(I) = R^n.$$

Thus, we have

$$\begin{aligned}V_n(R) &= \int_{D_R^n} 1\,dV \\ &= \int_{D_1^n} R^n\,dV \\ &= R^n\int_{D_1^n} 1\,dV \\ &= R^n V_n(1) \\ &= \alpha_n R^n\end{aligned}$$

as required.

(c) We have

$$\begin{aligned}V_n(R) &= \int_{-R}^{R}\int_{-\sqrt{R^2-x_1^2}}^{\sqrt{R^2-x_1^2}}\int_{-\sqrt{R^2-x_1^2-x_2^2}}^{\sqrt{R^2-x_1^2-x_2^2}}\cdots\int_{-\sqrt{R^2-x_1^2-x_2^2-\ldots-x_{n-1}^2}}^{\sqrt{R^2-x_1^2-x_2^2-\ldots-x_{n-1}^2}} dx_n \ldots dx_3\,dx_2\,dx_1 \\ &= \int_{D_R^2}\int_{D^{n-2}_{\sqrt{R^2-x_1^2-x_2^2}}} dV\,dA \\ &= \int_{D_R^2} V_{n-2}\left(\sqrt{R^2-x^2-y^2}\right)dA.\end{aligned}$$

as required.

$$= R^n \int_{D_1^n} 1 \, dV$$

$$\det DT = R^n \det I = R^n$$

$$= R^n V_n(1) \quad \boxed{\times}$$

c) $n \geq 3$

$$V_n(R) = \int_{D_R^2} V_{n-2}\left(\sqrt{R^2 - x^2 - y^2}\right) dA$$

$$= \int_0^{2\pi} \int_0^R V_{n-2}\left(\sqrt{R^2 - r^2}\right) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^R V_{n-2}(1) \left(R^2 - r^2\right)^{\frac{n-2}{2}} r \, dr \, d\theta$$

$$= \frac{V_{n-2}(1)}{-2} \int_0^{2\pi} \int_{R^2}^0 \left(u \right)^{\frac{n}{2} - 1} du \, d\theta$$

$u = R^2 - r^2$

$du = -2r \, dr$

$dr = \frac{du}{-2r}$

We have

$$\frac{V_{n-2}(1)}{-n}\int_0^{2\pi}\left[u^{n/2}\right]_{u=R^2}^{u=0} d\theta$$

$$\frac{V_{n-2}(1)}{-n}\int_0^{2\pi} -(R^2)^{n/2}\, d\theta$$

$$\frac{V_{n-2}(1)}{-n}\int_0^{2\pi} -R^n\, d\theta$$

$$\frac{V_{n-2}(1)R^n}{n}\int_0^{2\pi} 1\, d\theta$$

$$\frac{2\pi V_{n-2}(1)R^n}{n} \quad \text{since} \quad V_{n-2}(1) = \frac{V_{n-2}(R)}{R^{n-2}}$$

$$\frac{2\pi \frac{V_{n-2}(R)}{R^{n-2}} R^n}{n} = \frac{2\pi V_{n-2}(R)R^2}{n}$$

$$= \frac{2\pi R^2}{n} V_{n-2}(R) \quad \square$$

Pt 2:

(d)

d) n even

$$\alpha_n = V_n(1) = \frac{\pi^{n/2}}{(n/2)!}$$

Pf: $V_n(1) = \frac{2\pi}{n} V_{n-2}(1)$

$$= \frac{2\pi}{n}\,\frac{2\pi}{n-2}\,V_{n-4}(1)$$

$$= \underbrace{\left(\frac{2}{n}\right)\pi \left(\frac{2}{n-2}\right)\pi \left(\frac{2}{n-4}\right)\pi \cdots \left(\frac{2}{2}\right)\pi}_{k \text{ terms}} \qquad k = \frac{n}{2}$$

Base case: $V_2(R) = \pi R^2$

$$\frac{\pi^{n/2}}{\left(\frac{n}{2}\right)\left(\frac{n-2}{2}\right)\left(\frac{n-4}{2}\right)\cdots\left(\frac{2}{2}\right)}$$

$$\alpha_n = V_n(1) = \frac{\pi^{n/2}}{(n/2)!}$$

$$V_n(R) = \frac{\pi^{n/2}}{(n/2)!} R^n \qquad \square$$

(e)

e) $\alpha_n = \dfrac{\pi^{n/2}}{(n/2)!}$

$$\lim_{n\to\infty} \frac{\pi^{n/2}}{(n/2)!} = 0 \quad \text{for even } n$$

$$0 \le \frac{\pi^{n/2}}{(n/2)!} \le \prod_{i=1}^{n/2} \frac{\pi}{i} = \frac{\pi\cdot\pi\cdot\pi}{1\cdot 2\cdot 3}\prod_{i=4}^{n/2}\frac{\pi}{i} \le \frac{\pi^3}{6}\prod_{i=4}^{n/2}\frac{\pi}{4} = \frac{\pi^3}{6}\left(\frac{\pi}{4}\right)^{\frac{n}{2}-3}$$

0 as $n\to\infty$; $\frac{\pi^3}{6}\left(\frac{\pi}{4}\right)^{\frac{n}{2}-3} \to 0$ as $n\to\infty$

Thus, by squeeze thm

$$\lim_{n\to\infty} \frac{\pi^{n/2}}{(n/2)!} = 0$$

f) $C^n = \{x\in\mathbb{R}^n \mid \|x\| \le 1\}$

$n\to\infty \quad (n/2)!$ $\qquad 0$

f) $C^n = \{x \in \mathbb{R}^n \mid |x_i| \le 1\}$

$n=1$: $\int_{-1}^{1} dx = 2$

$n=2$: $\int_{-1}^{1}\int_{-1}^{1} 1 \, dx_1 \, dx_2 = 4$

$n=n$: $\int_{-1}^{1} \cdots \int_{-1}^{1}\int_{-1}^{1} 1 \, dx_1 \, dx_2 \cdots dx_{n-1} \, dx_n$

$\ldots \; 2\int_{-1}^{1} 1 \, dx_2$

2^n

As $n\to\infty$, volume $\to \infty$

(f)

MATH 222 — Homework #10

Due April 13, 2023

Maxwell Lin

Problem 1

Section 6.2, #8
Define $T(u,v) = \big(u^2 - v^2, 2uv\big)$. Let D^* be the set of (u,v) with $u^2+v^2 \le 1, u \ge 0, v \ge 0$. Find $T(D^*) = D$. Evaluate $\int\int_D dx\,dy$.

Solution

We can transform D^* into polar coordinates with the transformation $(u,v) = P(r,\theta) = (r\cos\theta, r\sin\theta)$. We obtain

$$u^2 + v^2 = r^2 \le 1$$
$$u = r\cos\theta \ge 0 \qquad v = r\sin\theta \ge 0$$

and thus,

$$D' = \left\{ (r,\theta) \;\middle|\; \begin{matrix} 0 \le r \le 1 \\ 0 \le \theta \le \dfrac{\pi}{2} \end{matrix} \right\}.$$

Therefore, to determine $T(D^*)$ we can equivalently solve for $T(P(D'))$ where D' is D^* expressed in polar coordinates. We have

$$\begin{aligned}(T \circ P) &= ((r\cos\theta)^2 - (r\sin\theta)^2, 2r^2\cos\theta\sin\theta) \\ &= r^2(\cos(2\theta), \sin(2\theta)).\end{aligned}$$

That is, $T(D^*) = (T \circ P)(D')$ is the closed region of radius 1 from angle 0 to π—the top half of the unit circle.

$$D = \left\{ (x,y) \;\middle|\; \begin{matrix} -1 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \end{matrix} \right\}$$

With this geometric picture, $\iint_D dx\,dy$ must be $\pi/2$. We evaluate

$$\det(\mathbf{D}T) = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix} = 4(u^2+v^2).$$

Since $\det(\mathbf{D}T) = 0$ only at $(0,0)$ which is on the boundary of D^*, we can change variables as follows

$$\begin{aligned}\iint_D dx\,dy &= \iint_{D^*} 4(u^2+v^2)\,du\,dv \\ &= \int_0^{\pi/2}\int_0^1 4r^2(r)\,dr\,d\theta \\ &= \frac{\pi}{2}\end{aligned}$$

as required.

Problem 2

Section 6.2, #21
Integrate $x^2 + y^2 + z^2$ over the cylinder $x^2 + y^2 \leq 2, -2 \leq z \leq 3$.

Solution

We use cylindrical coordinates as follows

$$\begin{aligned}
\int_D x^2 + y^2 + z^2 \, dV &= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{-2}^{3} (r^2 + z^2) r \, dz \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} 5r^3 + \frac{35}{3} r \, dr \, d\theta \\
&= \int_0^{2\pi} \frac{50}{3} d\theta \\
&= \frac{100\pi}{3}.
\end{aligned}$$

Problem 3

Section 6.2, #26
Use spherical coordinates to evaluate:

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} \frac{\sqrt{x^2 + y^2 + z^2}}{1 + \left[x^2 + y^2 + z^2\right]^2} \, dz \, dy \, dx$$

Solution

$$\begin{aligned}
\int_D f(x, y, z) \, dz \, dy \, dx &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 \frac{\rho}{1 + \rho^4} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 \frac{\rho^3 \sin\phi}{1 + \rho^4} \, d\rho \, d\phi \, d\theta \\
&= \int_0^{\pi/2} \int_0^{\pi/2} \frac{\sin\phi}{4} \int_1^{82} \frac{1}{u} \, du \, d\phi \, d\theta \qquad \text{substituting } u = 1 + \rho^4 \\
&= \int_0^{\pi/2} \frac{\ln(82)}{4} \int_0^{\pi/2} (\sin\phi) \, d\phi \, d\theta \\
&= \int_0^{\pi/2} \frac{\ln 82}{4} \, d\theta \\
&= \frac{\pi \ln 82}{8}.
\end{aligned}$$

Problem 4

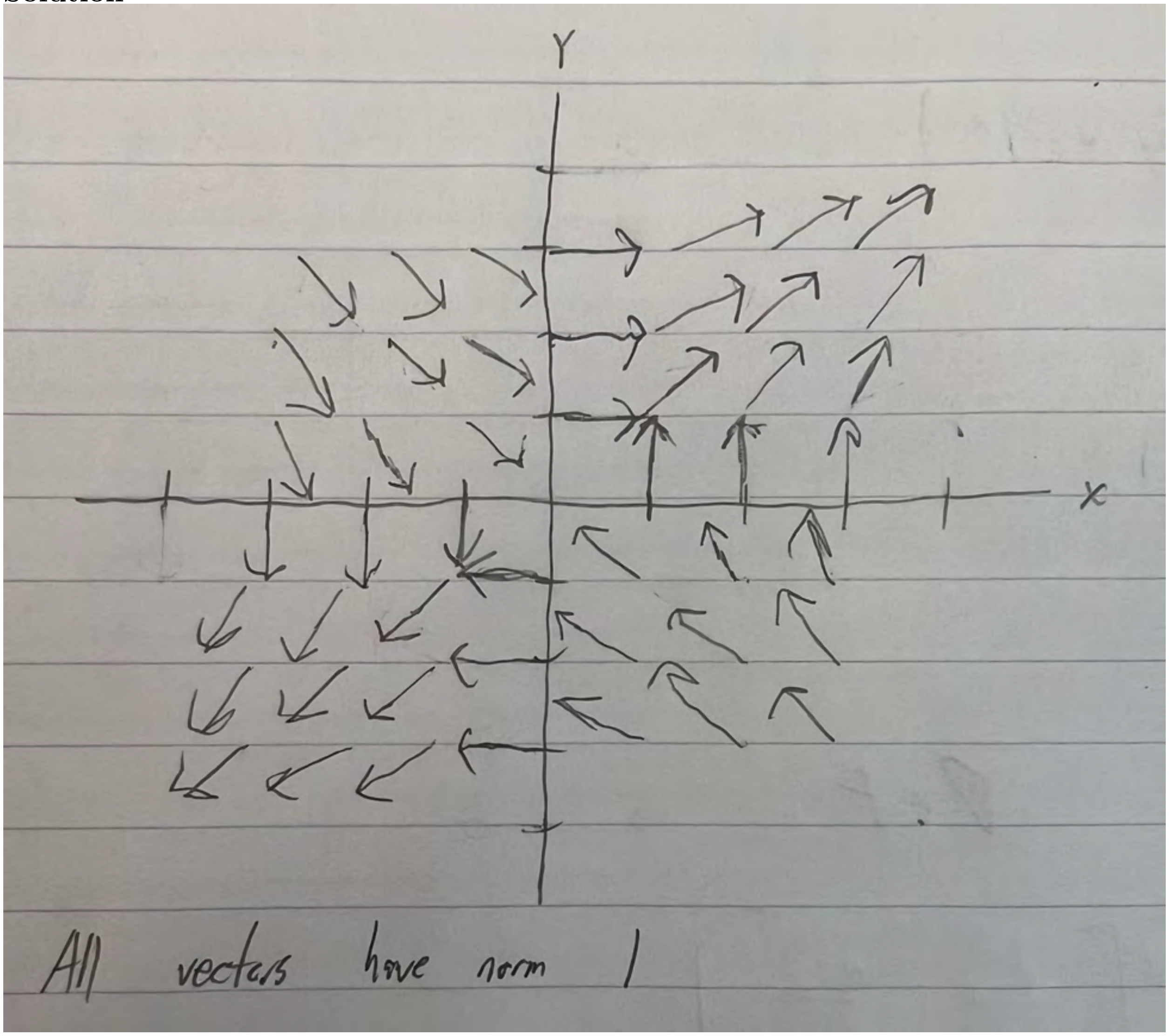
Section 4.3, #8

Sketch the given vector field or a small multiple of it.

$$\mathbf{F}(x, y) = \left(\frac{y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}} \right) \tag{1}$$

Problem 4 continued on next page. . .

Solution



Problem 5

Section 4.3, #17

Show that the given curve $c(t)$ is a flow line of the given velocity vector field $F(x, y, z)$.

$$\mathbf{c}(t) = \left(\sin t, \cos t, e^t\right); \mathbf{F}(x, y, z) = (y, -x, z) \tag{2}$$

Solution

We have that

$$c'(t) = (\cos t, -\sin t, e^t) = F(c(t)).$$

Thus, $c(t)$ is a flow line of $F(x, y, z)$.

Problem 6

Section 4.3, #24

Let $\mathbf{c}(t)$ be a flow line of a gradient field $\mathbf{F} = -\nabla V$ where V is a C^1 function $\mathbb{R}^n \to \mathbb{R}$. Prove that $V(\mathbf{c}(t))$ is a decreasing function of t.

Solution

Since $c(t)$ is a flow line of $F = -\nabla V$,

$$c'(t) = F(c(t)) = -\nabla V(c(t)).$$

To prove that $V(\mathbf{c}(t))$ is a decreasing function of t, we must show that $\mathbf{D}(V \circ c)(t) \leq 0$ for all t.

We have

$$\mathbf{D}(V \circ c)(t) = \nabla V(c(t)) \cdot c'(t) = \nabla V(c(t)) \cdot -\nabla V(c(t)) = -\|\nabla V(c(t))\|^2 \leq 0$$

as required.

Problem 7

Section 4.4, #4

Find the divergence of the vector field.

$$\mathbf{V}(x, y, z) = x^2\mathbf{i} + (x+y)^2\mathbf{j} + (x+y+z)^2\mathbf{k} \tag{3}$$

Solution

We have

$$\begin{aligned} \operatorname{div}(V) &= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \\ &= 2x + 2(x+y) + 2(x+y+z) \\ &= 6x + 4y + 2z. \end{aligned}$$

Problem 8

Section 4.4, #19

Calculate the scalar curl of the vector field.

$$\mathbf{F}(x, y) = xy\mathbf{i} + \left(x^2 - y^2\right)\mathbf{j} \tag{4}$$

Solution

We have

$$\begin{aligned} \operatorname{curl}(F) &= \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \\ &= 2x - x \\ &= x. \end{aligned}$$

Problem 9

Section 4.4, #25

Suppose $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^2 vector field. Which of the following expressions are meaningful, and which are nonsense? For those which are meaningful, decide whether the expression defines a scalar function or a vector field.

(a) curl(grad $\mathbf{F}$)
(b) grad(curl $\mathbf{F}$))
(c) div(grad $\mathbf{F}$)
(d) grad(div $\mathbf{F}$)
(e) curl(div $\mathbf{F}$)
(f) div(curl $\mathbf{F}$)

Solution

(a) Nonsense; cannot take gradient of vector-valued function.

(b) Nonsense; cannot take gradient of vector-valued function.

(c) Nonsense; cannot take gradient of vector-valued function.

(d) Meaningful; vector field.

(e) Nonsense; cannot take curl of scalar-valued function.

(f) Meaningful; scalar function.

Problem 10

Section 4.4, #27

Suppose $f, g, h : \mathbb{R}^2 \to \mathbb{R}$ are differentiable. Show that the vector field $\mathbf{F}(x, y, z) = (f(y, z), g(x, z), h(x, y))$ has zero divergence.

Solution

We have

$$\begin{aligned}
\operatorname{div}(F) &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\
&= 0 + 0 + 0 \qquad \text{since each } F_i \text{ does not depend on } x_i \\
&= 0.
\end{aligned}$$

MATH 222 — Homework #11

Due April 26, 2023

Maxwell Lin

Problem 1

Section 7.2: #4
Evaluate each of the following line integrals:
(a) $\int_{\mathbf{c}} xdy - ydx, \quad \mathbf{c}(t) = (\cos t, \sin t)\ 0 \le t \le 2\pi$
(b) $\int_{\mathbf{c}} xdx + ydy, \quad \mathbf{c}(t) = (\cos \pi t, \sin \pi t),\ 0 \le t \le 2$
(c) $\int_{\mathbf{c}} yzdx + xzdy + xydz$, where c consists of straight-line segments joining $(1,0,0)$ to $(0,1,0)$ to $(0,0,1)$
(d) $\int_{\mathbf{c}} x^2dx - xydy + dz$, where $\mathbf{c}$ is the parabola $z = x^2, y = 0$ from $(-1,0,1)$ to $(1,0,1)$

Solution

(a) We have

$$\begin{aligned} x &= \cos t & dx &= -\sin t\, dt \\ y &= \sin t & dy &= \cos t\, dt. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\mathbf{c}} xdy - ydx &= \int_0^{2\pi} \cos^2(t) + \sin^2(t)\, dt \\ &= 2\pi. \end{aligned}$$

(b) We have

$$\begin{aligned} \int_{\mathbf{c}} xdx + ydy &= \int_0^2 -\pi\cos(\pi t)\sin(\pi t) + \pi\sin(\pi t)\cos(\pi t)\, dt \\ &= 0. \end{aligned}$$

(c) We have

$$\int_{\mathbf{c}} yzdx + xzdy + xydz = \int_{c_1} yzdx + xzdy + xydz + \int_{c_2} yzdx + xzdy + xydz$$

where c_1 parameterizes the straight-line segment joining $(1,0,0)$ to $(0,1,0)$ and c_2 parameterizes the straight-line-segment joining $(0,1,0)$ to $(0,0,1)$.

We obtain

$$\begin{aligned} c_1 &= (1-t, t, 0) \qquad t \in [0,1] \\ c_2 &= (0, 1-t, t) \qquad t \in [0,1]. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\mathbf{c}} yzdx + xzdy + xydz &= \int_{c_1} 0+0+0 + \int_{c_2} 0+0+0 \\ &= 0. \end{aligned}$$

(d) We parameterize c with

$$c(t) = (t, 0, t^2) \qquad t \in [-1, 1].$$

Thus,

$$\begin{aligned}\int_{\mathbf{c}} x^2 dx - xydy + dz &= \int_{-1}^{1} t^2 + 2t\, dt \\ &= \left[\frac{t^3}{3} + t^2\right]_{-1}^{1} \\ &= \frac{2}{3}.\end{aligned}$$

Problem 2

Section 7.2: #12

Suppose $\mathbf{c}_1$ and $\mathbf{c}_2$ are two paths with the same endpoints and $\mathbf{F}$ is a vector field. Show that $\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$ is equivalent to $\int_{c} \mathbf{F} \cdot d\mathbf{s} = 0$, where C is the closed curve obtained by first moving along $\mathbf{c}_1$ and then moving along $\mathbf{c}_2$ in the opposite direction.

Solution

Since the oriented curve C can be obtained by moving along the curve parameterized by c_1 then moving along the curve parameterized by c_2 in the opposite orientation,

$$\int_C F \cdot ds = \int_{c_1} F \cdot ds - \int_{c_2} F \cdot ds = 0.$$

This is equivalent to

$$\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}.$$

Problem 3

Section 7.3: #8

Match the following parametrizations to the surfaces shown in the figures.

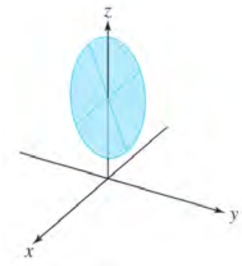
(a) $\Phi(u, v) = (u \cos v, u \sin v, 4 - u \cos v - u \sin v)$; $u \in [0, 1], v \in [0, 2\pi]$

(b) $\Phi(u, v) = \big(u \cos v, u \sin v, 4 - u^2\big)$

(c) $\Phi(u, v) = \big(u, v, \frac{1}{3}(12 - 8u - 3v)\big)$

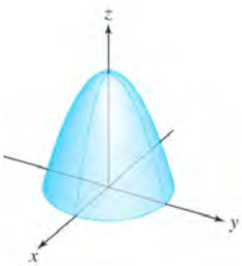
(d) $\Phi(u, v) = \big(\big(u^2 + 6u + 11\big) \cos v\,,\ u, \big(u^2 + 6u + 11\big) \sin v\big)$

Problem 3 continued on next page. . .

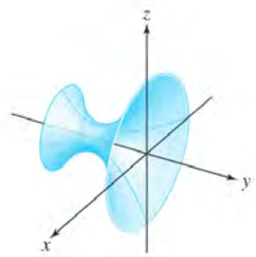


(i)

(ii)



(iii)



(iv)

Solution

(a) (i)

(b) (iii)

(c) (ii)

(d) (iv)

Problem 4

Section 7.3: #23
The image of the parametrization

$$\begin{aligned}\Phi(u,v) &= (x(u,v), y(u,v), z(u,v))\\ &= ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)\end{aligned}$$

with $0 \le u, v \le 2\pi, 0 < r < 1, R > 1$ parametrizes a torus (or doughnut) S.
(a) Show that all points in the image (x, y, z) satisfy:

$$\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2.$$

(b) Show that the image surface is regular at all points.

Solution

(a) We have

$$\begin{aligned}\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 &= \left(\sqrt{(R + r\cos u)^2\cos^2 v + (R + r\cos u)^2\sin^2 v} - R\right)^2 + r^2\sin^2 u\\ &= r^2\cos^2 u + r^2\sin^2 u\\ &= r^2\end{aligned}$$

as required.

(b) To show that the image surface is regular at all points, we must show that $T_u \times T_v \neq 0$.

We have

$$T_u = \begin{bmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{bmatrix} \qquad T_v = \begin{bmatrix} -(R+r\cos u)\sin v \\ (R+r\cos u)\cos v \\ 0 \end{bmatrix}$$

$$T_u \times T_v = -r(R+r\cos u)\begin{bmatrix} \cos u\cos v \\ \cos u\sin v \\ \sin u. \end{bmatrix}$$

By assumption, $r \neq 0$. Additionally, $R + r\cos u \neq 0$ since $-1 < r\cos u < 1$ and $R > 1$. Therefore, we must prove that

$$A = \begin{bmatrix} \cos u\cos v \\ \cos u\sin v \\ \sin u \end{bmatrix} \neq 0.$$

For the sake of contradiction, assume that $A = 0$. Then, $\sin u = 0$ which means $u = k\pi$ for $k \in \mathbb{Z}^+ \cup \{0\}$. We also must have that

$$\cos u\cos v = \cos u\sin v = 0$$
$$\cos v = \sin v = 0 \qquad \text{since } \cos u \neq 0.$$

However, this equation has no solutions. Therefore, $A \neq 0$ and the image surface must be regular at all points.

Problem 5

Section 7.4: #1

Find the surface area of the unit sphere S represented parametrically by $\mathbf{\Phi} : D \to S \subset \mathbb{R}^3$, where D is the rectangle $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$ and $\mathbf{\Phi}$ is given by the equations

$$x = \cos\theta\sin\phi, \quad y = \sin\theta\sin\phi, \quad z = \cos\phi$$

Note that we can represent the entire sphere parametrically, but we cannot represent it in the form $z = f(x, y)$

Solution

The formula for surface area is

$$\iint_D \|T_\theta \times T_\phi\| \, dA.$$

We have

$$T_\theta = \begin{bmatrix} -\sin\phi\sin\theta \\ \sin\phi\cos\theta \\ 0 \end{bmatrix} \qquad T_\phi = \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ -\sin\phi \end{bmatrix}$$

$$T_\theta \times T_\phi = \begin{bmatrix} -\sin^2\phi\cos\theta \\ -\sin^2\phi\sin\theta \\ -\sin\phi\cos\phi \end{bmatrix}$$

Problem 5 continued on next page. . .

$$\begin{aligned}\|T_\theta \times T_\phi\| &= \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi} \\ &= \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} \\ &= \sqrt{\sin^2\phi} \\ &= \sin\phi \qquad \text{since } \phi \in [0,\pi].\end{aligned}$$

Therefore,

$$\begin{aligned}\iint_D \|T_\theta \times T_\phi\|\, dA &= \int_0^\pi \int_0^{2\pi} (\sin\phi)\, d\theta\, d\phi \\ &= 4\pi.\end{aligned}$$

Problem 6

Section 7.4: #4
The torus T can be represented parametrically by the function $\mathbf{\Phi} : D \to \mathbb{R}^3$, where $\mathbf{\Phi}$ is given by the coordinate functions $x = (R + \cos\phi)\cos\theta$ $y = (R + \cos\phi)\sin\theta, z = \sin\phi; D$ is the rectangle $[0, 2\pi] \times [0, 2\pi]$, that is, $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq 2\pi$; and $R > 1$ is fixed (see Figure 7.4.6). Show that $A(T) = (2\pi)^2 R$, first by using formula (3) and then by using formula (6).

Solution

Formula (3) is

$$A(S) = \iint_D \sqrt{\left[\frac{\partial(x,y)}{\partial(\theta,\phi)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(\theta,\phi)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(\theta,\phi)}\right]^2}\, d\theta\, d\phi.$$

We compute as follows

$$\begin{aligned}\frac{\partial(x,y)}{\partial(\theta,\phi)} &= \begin{vmatrix} -(R+\cos\phi)\sin\theta & -\sin\phi\cos\theta \\ (R+\cos\phi)\cos\theta & -\sin\phi\sin\theta \end{vmatrix} \\ &= (R+\cos\phi)\sin\phi\end{aligned}$$

$$\begin{aligned}\frac{\partial(y,z)}{\partial(\theta,\phi)} &= \begin{vmatrix} (R+\cos\phi)\cos\theta & -\sin\phi\sin\theta \\ 0 & \cos\phi \end{vmatrix} \\ &= (R+\cos\phi)\cos\theta\cos\phi\end{aligned}$$

$$\begin{aligned}\frac{\partial(x,y)}{\partial(\theta,\phi)} &= \begin{vmatrix} -(R+\cos\phi)\sin\theta & -\sin\phi\cos\theta \\ 0 & \cos\phi \end{vmatrix} \\ &= -(R+\cos\phi)\sin\theta\cos\phi\end{aligned}$$

$$\sqrt{\left[\frac{\partial(x,y)}{\partial(\theta,\phi)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(\theta,\phi)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(\theta,\phi)}\right]^2} = R + \cos\phi.$$

Thus, we have

$$\begin{aligned}A(s) &= \int_0^{2\pi} \int_0^{2\pi} (R + \cos\phi)\, d\phi\, d\theta \\ &= \int_0^{2\pi} 2\pi R\, d\theta \\ &= (2\pi)^2 R\end{aligned}$$

Problem 6 continued on next page...

as required.

Formula (6) is

$$A = 2\pi \int_a^b \left(|x|\sqrt{1 + [f'(x)]^2}\right) dx$$

We have

$$f(x) = \sqrt{1 - (x-R)^2}$$

which is the graph of the upper half cross-section of the torus we wish to revolve about the y-axis.
We compute

$$f'(x) = \frac{-(x-R)}{\sqrt{1-(x-R)^2}}$$

and

$$\begin{aligned}\sqrt{1+[f'(x)]^2} &= \sqrt{1 + \frac{(x-R)^2}{1-(x-R)^2}} \\ &= \frac{1}{\sqrt{1-(x-R)^2}}.\end{aligned}$$

Thus, we have

$$\begin{aligned}2\pi \int_a^b \left(|x|\sqrt{1+[f'(x)]^2}\right) dx &= 2\pi \int_{R-1}^{R+1} \frac{x}{\sqrt{1-(x-R)^2}}\, dx \qquad |x| = x \text{ since } R > 1 \\ &= 2\pi \left[\int_{R-1}^{R+1} \frac{x-R}{\sqrt{1-(x-R)^2}}\, dx + \int_{R-1}^{R+1} \frac{R}{\sqrt{1-(x-R)^2}}\, dx\right]\end{aligned}$$

We compute the first integral

$$\begin{aligned}\int_{R-1}^{R+1} \frac{x-R}{\sqrt{1-(x-R)^2}}\, dx &= \int_0^0 \frac{du}{2\sqrt{u}} \qquad \text{substitute } u = 1-(x-R)^2 \\ &= 0.\end{aligned}$$

We compute the second integral

$$\begin{aligned}\int_{R-1}^{R+1} \frac{R}{\sqrt{1-(x-R)^2}}\, dx &= [R\arcsin(x-R)]_{R-1}^{R+1} \\ &= \pi R.\end{aligned}$$

Thus,

$$2\pi \int_a^b \left(|x|\sqrt{1+[f'(x)]^2}\right) dx = 2\pi^2 R.$$

Multiplying by 2 (since we can only graph half of the torus cross-section), we obtain $4\pi^2 R$ as required.

Problem 7

Section 7.5: #9
Evaluate $\iint_S z dS$, where S is the upper hemisphere of radius a, that is, the set of (x, y, z) with $z = \sqrt{a^2 - x^2 - y^2}$

Solution

Using spherical coordinates, we obtain the surface parametrization

$$\Phi(\theta, \phi) = (a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi)$$

where $\phi \in [0, \pi/2]$ and $\theta \in [0, 2\pi]$.

We have

$$T_\theta = \begin{bmatrix} -a \sin\phi \sin\theta \\ a \sin\phi \cos\theta \\ 0 \end{bmatrix} \qquad T_\phi = \begin{bmatrix} a \cos\phi \cos\theta \\ a \cos\phi \sin\theta \\ -a \sin\phi \end{bmatrix}$$

$$T_\theta \times T_\phi = \begin{bmatrix} -a^2 \sin^2\phi \cos\theta \\ -a^2 \sin^2\phi \sin\theta \\ -a^2 \cos\phi \sin\phi \end{bmatrix}$$

$$\|T_\theta \times T_\phi\| = a^2 \sin\phi.$$

Thus, we obtain

$$\begin{aligned} \iint_S z dS &= \int_0^{2\pi} \int_0^{\pi/2} (a \cos\phi)(a^2 \sin\phi) \, d\phi \, d\theta \\ &= \int_0^{2\pi} \frac{a^3}{2} \, d\theta \\ &= \pi a^3. \end{aligned}$$

Problem 8

Section 7.5: #12
Verify that in spherical coordinates, on a sphere of radius R,

$$\|\mathbf{T}_\phi \times \mathbf{T}_\theta\| \, d\phi \, d\theta = R^2 \sin\phi \, d\phi \, d\theta$$

Solution

We have the surface parametrization

$$\Phi(\theta, \phi) = (R \sin\phi \cos\theta, R \sin\phi \sin\theta, R \cos\phi)$$

where $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi]$.

We compute

$$T_\theta = \begin{bmatrix} -R \sin\phi \sin\theta \\ R \sin\phi \cos\theta \\ 0 \end{bmatrix} \qquad T_\phi = \begin{bmatrix} R \cos\phi \cos\theta \\ R \cos\phi \sin\theta \\ -R \sin\phi \end{bmatrix}$$

$$T_\theta \times T_\phi = \begin{bmatrix} -R^2 \sin^2\phi \cos\theta \\ -R^2 \sin^2\phi \sin\theta \\ -R^2 \cos\phi \sin\phi \end{bmatrix}$$

Problem 8 continued on next page...

$$\begin{aligned}
\|T_\theta \times T_\phi\| &= \sqrt{(-R^2\sin^2\phi\cos\theta)^2 + (-R^2\sin^2\phi\sin\theta)^2 + (-R^2\cos\phi\sin\phi)^2} \\
&= \sqrt{R^4\sin^2\phi[\sin^2\phi(\cos^2\theta+\sin^2\theta)+\cos^2\phi]} \\
&= \sqrt{R^4\sin^2\phi} \\
&= R^2\sin\phi \qquad \text{since } R^2\sin\phi \geq 0
\end{aligned}$$

as required.

Problem 9

Section 7.6: #2
Evaluate the surface integral

$$\iint_S \mathbf{F}\cdot d\mathbf{S}$$

where $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$ and S is the surface parameterized by $\Phi(u,v) = (2\sin u, 3\cos u, v)$, with $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$

Solution

We have

$$T_u = \begin{bmatrix} 2\cos u \\ -3\sin u \\ 0 \end{bmatrix} \qquad T_v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T_u \times T_v = \begin{bmatrix} -3\sin u \\ -2\cos u \\ 0 \end{bmatrix}$$

$$(T_u \times T_v)\cdot F = \begin{bmatrix} -3\sin u \\ -2\cos u \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2\sin u \\ 3\cos u \\ v^2 \end{bmatrix} = -6.$$

Thus, we obtain

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^1 -6\,dv\,du \\
&= -12\pi.
\end{aligned}$$

Problem 10

Section 7.6: #7
Let S be the closed surface that consists of the hemisphere $x^2+y^2+z^2=1, z\geq 0$, and its base $x^2+y^2 \leq 1, z=0$. Let $\mathbf{E}$ be the electric field defined by $\mathbf{E}(x,y,z) = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$. Find the electric flux across S. (HINT: Break S into two pieces S_1 and S_2 and evaluate $\iint_{S_1}\mathbf{E}\cdot d\mathbf{S}$ and $\iint_{S_2}\mathbf{E}\cdot d\mathbf{S}$ separately.)

Solution

The electric flux across S is

$$\iint_S E\cdot\, dS = \iint_{S_1} E\cdot\, dS + \iint_{S_2} E\cdot\, dS$$

Problem 10 continued on next page. . .

where S_1 is the hemisphere and S_2 is the base, both with unit normals that face outwards. We parametrize S_1 with

$$\Phi_1(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

where $\phi \in [0, \pi/2]$ and $\theta \in [0, 2\pi]$.

From Problem 7, we know that

$$T_\phi \times T_\theta = \begin{bmatrix} \sin^2\phi\cos\theta \\ \sin^2\phi\sin\theta \\ \cos\phi\sin\phi \end{bmatrix}.$$

Thus,

$$\begin{aligned} E \cdot (T_\phi \times T_\theta) &= \begin{bmatrix} 2\sin\phi\cos\theta \\ 2\sin\phi\sin\theta \\ 2\cos\phi \end{bmatrix} \cdot \begin{bmatrix} \sin^2\phi\cos\theta \\ \sin^2\phi\sin\theta \\ \cos\phi\sin\phi \end{bmatrix} \\ &= 2\sin\phi. \end{aligned}$$

$$\begin{aligned} \iint_{S_1} E \cdot dS &= \int_0^{2\pi} \int_0^{\pi/2} (2\sin\phi)\, d\phi\, d\theta \\ &= 4\pi. \end{aligned}$$

Now we parameterize S_2 by

$$\Phi_2(r, \theta) = (r\cos\theta, r\sin\theta, 0)$$

where $r \in [0, 1]$ and $\theta \in [0, 2\pi]$. We obtain

$$T_r = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \qquad T_\theta = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix}$$

$$T_\theta \times T_r = \begin{bmatrix} 0 \\ 0 \\ -r \end{bmatrix}.$$

Thus,

$$\begin{aligned} E \cdot (T_\theta \times T_r) &= \begin{bmatrix} 2r\cos\theta \\ 2r\sin\theta \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -r \end{bmatrix} \\ &= 0. \end{aligned}$$

Therefore,

$$\iint_{S_2} E \cdot dS = 0$$

and

$$\iint_S E \cdot dS = 4\pi.$$